

AN APPLICATION OF SYMBOLIC METHODS TO THE TREATMENT OF MEAN CURVATURES IN HYPERSPACE*

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This paper is an application of MASCHKE's symbolic method for discussing invariants of quadratic differential forms, as developed in his article, *A Symbolic Treatment of the Theory of Invariants of Quadratic Differential Quantics of n Variables*.† Extensive use is also made of results and methods contained in two later publications, *Differential Parameters of the First Order*,‡ and *The Kronecker-Gaussian Curvature of Hyperspace*.§ Some familiarity with these three articles is implied.

Part I of the present paper is devoted to the study of the curvatures of an n -space R_n in an euclidean $(n + 1)$ -space S_{n+1} . In §§ 1–3 the equations and some of the properties of the lines of curvature of R_n in S_{n+1} are developed. In particular, equation (28) gives the n curvatures of the n lines of curvature through a given point of R_n . The coefficients K_1, \dots, K_n of this equation are the so-called curvatures of R_n in S_{n+1} , involving the coefficients a_{ik} and α_{ik} of the two fundamental forms of R_n . With the help of his symbolic method,|| MASCHKE has expressed K_n , when n is even, and K_n^2 , when n is odd, as rational integral functions of the coefficients a_{ik} of the first fundamental form and their derivatives.

In §§ 4–6 similar expressions are derived for all the curvatures $K_{2\nu}$ of even index. It does not seem possible to obtain rational results for the curvatures $K_{2\nu+1}$ of odd index. In § 7, however, it is shown that, with the exception of K_1 , these curvatures are expressible irrationally in terms of the first fundamental quantities and their derivatives.

The symbolic expressions for $K_{2\nu}$ and K_n^2 show at once that they are differential invariants of the first fundamental quadratic form for R_n , and they have meaning as invariants of any quadratic form in n variables. Part II of this paper considers a space R_λ defined in a space R_n ($n > \lambda$), which is not neces-

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† These Transactions, vol. 4 (1903), pp. 445–469. This paper is referred to hereafter as M. I.

‡ *Ibid.*, vol. 7 (1906), pp. 69–80; referred to as D. P.

§ *Ibid.*, vol. 7 (1906), pp. 81–93; referred to as K.–G. C.

|| In K.–G. C.

sarily euclidean. The invariants $K_{2\nu}$ and K_λ^2 for R_λ are calculated in terms of the coefficients a_{ik} belonging to the length element of R_n and of the functions $U^{\lambda+1}, \dots, U^n$ which determine the space R_λ in R_n .

PART I.

CURVATURES OF AN n -SPACE IN AN $(n+1)$ -SPACE.

§ 1. *Parametric Representation for an n -space in an $(n+1)$ -space.*

Let z', z^2, \dots, z^{n+1} be the coördinates* of an euclidean space S_{n+1} of $n+1$ dimensions, i. e., a space whose arc-element is of the form

$$(1) \quad ds^2 = \sum_{i=1}^{n+1} [dz^i]^2.$$

We define in S_{n+1} any hypersurface, or space R_n , of n dimensions, by expressing each z as a function of n independent variables x_1, \dots, x_n :

$$(2) \quad \begin{aligned} z' &= z'(x_1, \dots, x_n), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ z^{n+1} &= z^{n+1}(x_1, \dots, x_n). \end{aligned}$$

The arc-element of R_n is given by the equation

$$(3) \quad ds^2 = \sum_{i,k}^{1, \dots, n} a_{ik} dx_i dx_k,$$

where

$$(4) \quad a_{ik} = \sum_{j=1}^{n+1} \frac{\partial z^j}{\partial x_i} \frac{\partial z^j}{\partial x_k} = \sum_{j=1}^{n+1} z_i^j z_k^j = f_i f_k,$$

differentiation with respect to x_i being indicated here, as in the following pages, by the lower index i .

A space of λ dimensions, $\lambda < n$, would be obtained by using in (2) only λ independent variables x_1, \dots, x_λ . In particular, a curve in S_{n+1} is obtained by expressing each z as a function of one new variable x .

If in (2) one puts $x_2 = \dots = x_n = 0$, the resulting curve is called the x_1 -axis of a curvilinear system of coördinates on R_n . By letting x_2, \dots, x_n represent arbitrary constants, one gets the complete system of x_1 -curves; and similarly for the other cases.

$$(5) \quad \begin{aligned} &x_1\text{-curves if } x_1 \text{ varies and } x_2, \dots, x_n \text{ are constants,} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &x_n\text{-curves if } x_n \text{ varies and } x_1, \dots, x_{n-1} \text{ are constants.} \end{aligned}$$

Equations (3) and (5) give the elements of the new axes,

$$(6) \quad ds_1^2 = a_{11} dx_1^2, \dots, ds_n^2 = a_{nn} dx_n^2,$$

*It is assumed that no confusion will arise from writing the upper index, as MASCHKE does, without parentheses. Exponents are only occasionally used, and will be easily recognized.

where ds_k is the element of the x_k -axis. Represent the direction cosines of the x_k -axis, in the old system, by $\cos(z', x_k), \dots, \cos(z^{n+1}, x_k)$. Then

$$(7) \quad \cos(z^j, x_k) = \frac{dz^j}{ds_k} = \frac{z_k^j dx_k}{\sqrt{a_{kk}} dx_k} = \frac{z_k^j}{\sqrt{a_{kk}}} \quad (j=1, \dots, n+1; k=1, \dots, n).$$

Let ω_{ik} be the angle between the x_i -axis and the x_k -axis. Then

$$(8) \quad \cos \omega_{ik} = \sum_{j=1}^{n+1} \cos(z^j x_i) \cos(z^j x_k) = \sum_{j=1}^{n+1} \frac{z_i^j z_k^j}{\sqrt{a_{ii}} \sqrt{a_{kk}}} = \frac{a_{ik}}{\sqrt{a_{ii}} \sqrt{a_{kk}}} \quad (i, k=1, \dots, n),$$

so that *necessary and sufficient conditions for mutual orthogonality of the axes of the new system are*

$$(9) \quad a_{ik} = 0 \quad (i, k=1, \dots, n; i \neq k).$$

§ 2. General Curves on R_n .

A general curve on R_n may be defined by means of $n-1$ equations,

$$(10) \quad U^2(x_1, \dots, x_n) = \text{const.}, \dots, U^n(x_1, \dots, x_n) = \text{const.}$$

The differential equations of this curve, which we call the U -curve, are

$$(11) \quad \sum_{i=1}^n U_i^2 dx_i = 0, \dots, \sum_{i=1}^n U_i^n dx_i = 0.$$

Its direction is defined by the ratios of dx_1, \dots, dx_n in (11). In order to solve for these differentials, let p be any function of x_1, \dots, x_n which satisfies the condition *

$$D = (pU^2 \dots U^n) = (pU) \neq 0.$$

If A' denotes the cofactor of p_r in D , equations (11) are identically satisfied by

$$(12) \quad dx_1 = \rho A', \dots, dx_n = \rho A^n,$$

where ρ is an arbitrary parameter.

The direction cosines ξ', \dots, ξ^{n+1} of the U -curve are found as follows. From (12),

$$(13) \quad \sum_{i=1}^n p_i dx_i = \rho \sum_{i=1}^n p_i A^i = \rho(pU).$$

Then

$$\xi^k = \frac{dz^k}{ds} = \frac{1}{ds} \sum_{i=1}^n z_i^k dx_i = \frac{\rho}{ds} (z^k U),$$

where ds is arc-element of the U -curve. Now

$$\sum_{k=1}^n [\xi^k]^2 = 1 = \sum_{k=1}^{n+1} \left[\frac{\rho}{ds} \right]^2 (z^k U)^2 = \left[\frac{\rho}{ds} \right]^2 (fU)^2.$$

* See M. I., § 2, for an explanation of this invariantive notation.

Hence

$$\frac{\rho}{ds} = \frac{1}{V(fU)^2}.$$

Then the direction cosines of the U -curve on R_n , referred to the original system of axes, are

$$(14) \quad \xi' = \frac{(z' U)}{V(fU)^2}, \quad \dots, \quad \xi^{n+1} = \frac{(z^{n+1} U)}{V(fU)^2}.$$

If there is given also a V -curve on R_n by equations similar to (10), its direction cosines may be written

$$(15) \quad \eta' = \frac{(z' V)}{V(fV)^2}, \quad \dots, \quad \eta^{n+1} = \frac{(z^{n+1} V)}{V(fV)^2}.$$

If ω is the angle between the two curves, we have from (14) and (15)

$$(16) \quad \cos \omega = \sum_{i=1}^{n+1} \xi^i \eta^i = \sum_{i=1}^{n+1} \frac{(z^i U)(z^i V)}{V(fU)^2 V(fV)^2} = \frac{(fU)(fV)}{V(fU)^2 V(fV)^2}.$$

Thus a necessary and sufficient condition for orthogonality of the two curves is

$$(17) \quad (fU)(fV) = 0.$$

Equation (17) also defines the orthogonal trajectories of a system of U -curves on R_n . An illustration is found in the case of curves on an ordinary surface.

§ 3. Lines of Curvature on R .

A line L drawn on R_n such that the normals to R_n along L (with respect to the enclosing space S_{n+1}) generate a developable surface is called* a line of curvature of R_n in S_{n+1} .

At a point P of R_n there is a unique normal to R_n in S_{n+1} . Let the direction cosines of this normal be $\zeta', \dots, \zeta^{n+1}$. Choose P as origin of the system of x -axes on R_n . Then, since the normal to R_n at P is orthogonal to every direction on R_n at P , we have from (7)

$$(18) \quad \sum_{i=1}^{n+1} \zeta^i z_k^i = 0 \quad (k = 1, \dots, n).$$

The coefficients α_{ik} of the first fundamental form of R_n , given in (3) are the first fundamental quantities. The second fundamental quantities are defined by the equations

$$\alpha_{ik} = \sum_{j=1}^{n+1} \zeta^j z_{ik}^j \quad (i, k = 1, \dots, n).$$

* Cf. BIANCHI, *Lezioni di Geometria Differenziale*, vol. I, p. 125.

If now a line of curvature be represented as a U -curve (10), one gets from (13) and (29)

$$(30) \quad f_k(fU) = -rg_k(gU) \quad (k=1, \dots, n).$$

A symmetrical expression for r is obtained by multiplying equations (30) in order by the cofactors of f_1, \dots, f_n in (fU) and adding:

$$(31) \quad r = -\frac{(fU)^2}{(gU)^2}.$$

If any two lines of curvature through P be given as U and V -curves, and their respective curvatures be denoted by $1/r'$ and $1/r''$, one gets from (30)

$$g_1(gU) = -\frac{1}{r'}f_1(fU), \dots, g_n(gU) = -\frac{1}{r'}f_n(fU),$$

$$g_1(gV) = -\frac{1}{r''}f_1(fV), \dots, g_n(gV) = -\frac{1}{r''}f_n(fV).$$

Multiply the equations of the first line in order by the cofactors of f_1, \dots, f_n in (fV) and add. Also multiply the equations of the second line in order by the cofactors of f_1, \dots, f_n in (fU) and add. Then

$$(gU)(gV) = -\frac{1}{r'}(fU)(fV) = -\frac{1}{r''}(fU)(fV),$$

so that either $r' = r''$ or $(fU)(fV) = 0$. Hence by (17) we have

Theorem I. *Any two distinct lines of curvature through an ordinary point P of R_n are orthogonal to each other.*

If the lines of curvature through P be taken as parameter lines, then, by (9),

$$\alpha_{ik} = 0 \quad (i \neq k).$$

It follows at once from (26) that also

$$\alpha_{ik} = 0 \quad (i, k=1, \dots, n; i \neq k).$$

Theorem II. *If the lines of curvature at an ordinary (not umbilic) point of R_n be taken as parameter lines, then*

$$\alpha_{ik} = 0, \quad \alpha_{ik} = 0 \quad (i, k=1, \dots, n; i \neq k).$$

§ 4. Definition of the Curvatures of R_n in S_{n+1} .

Equation (28) may be written in the form

$$(32) \quad H_0 + H_1r + \dots + H_{n-1}r^{n-1} + H_nr^n = 0,$$

where*

$$H_0 = |\alpha_{ik}| = 1/\beta^2, \quad H_n = |\alpha_{ik}|,$$

* M. I., (9).

while for $j = 1, \dots, n$, H_j is the sum of all the determinants obtained from $|a_{ik}|$ by replacing in all possible ways j columns of $|a_{ik}|$ by the corresponding columns of $|\alpha_{ik}|$. Dividing (32) by H_0 one obtains

$$(33) \quad 1 + K_1 r + \dots + K_{n-1} r^{n-1} + K_n r^n = 0.$$

The coefficient K_n (the product of all the curvatures) is the Kronecker-Gaussian curvature of hyperspace. It has been shown to be expressible in terms of the first fundamental quantities and their derivatives (cf. K-G. C.). In this paper the coefficients of (33) are called the n curvatures of R_n in S_{n+1} . By definition

$$(34) \quad K_1 = \beta^2 \sum_{i,k}^{1,\dots,n} \alpha_{ik} A_k^i, \quad K_2 = \beta^2 \sum_{i_1 i_2, k_1 k_2}^{1,\dots,n} \begin{vmatrix} \alpha_{i_1 k_1} & \alpha_{i_1 k_2} \\ \alpha_{i_2 k_1} & \alpha_{i_2 k_2} \end{vmatrix} \cdot A_{k_1 k_2}^{i_1 i_2} = \sum_{i_1 i_2, k_1 k_2}^{1,\dots,n} \Delta_{k_2 k_2}^{i_1 i_2} A_{k_1 k_2}^{i_1 i_2}, \dots,$$

$$K_m = \beta^2 \sum_{i_1 \dots i_m, k_1 \dots k_m}^{1,\dots,n} \Delta_{k_1 \dots k_m}^{i_1 \dots i_m} \cdot A_{k_1 \dots k_m}^{i_1 \dots i_m} \quad (m = 1, \dots, n),$$

where A_k^i is the cofactor of a_{ik} in $|a_{ik}|$, $A_{k_1 k_2}^{i_1 i_2}$ is the *algebraic complement* of

$$\begin{vmatrix} a_{i_1 k_1} & a_{i_1 k_2} \\ a_{i_2 k_1} & a_{i_2 k_2} \end{vmatrix}$$

in $|a_{ik}|$, while $\Delta_{k_1 k_2}^{i_1 i_2}$ is the *second minor* of $|a_{ik}|$ indicated for K_2 above; and similarly for the A 's and Δ 's in K_m . Both sets i_1, \dots, i_m and k_1, \dots, k_m are considered as being in ascending numerical order.

§ 5. Invariant Symbolic Forms of K_1, \dots, K_n .

If F_k^i be the cofactor of f_k^i in the functional determinant $\{f', \dots, f^n\}$, Maschke* has shown that

$$A_k^i = \frac{1}{(n-1)!} F_i' F_k'.$$

Thus

$$\begin{aligned} K_1 &= \beta^2 \sum_{i,k}^{1,\dots,n} \alpha_{ik} A_k^i = \frac{\beta^2}{(n-1)!} \sum_{i,k}^{1,\dots,n} g_i g_k F_i' F_k' \\ &= \frac{\beta^2}{(n-1)!} \{g f^2 \dots f^n\}^2 = \frac{1}{(n-1)!} (g f)^2. \end{aligned}$$

This suggests a method for reducing all the curvatures to convenient invariant forms. Let $F_{i_1 \dots i_m}^{1 \dots m}$ be the algebraic complement of

$$\begin{vmatrix} f_{i_1}' & \dots & f_{i_m}' \\ \dots & \dots & \dots \\ f_{i_1}^m & \dots & f_{i_m}^m \end{vmatrix}$$

* M. I., p. 450.

This gives Maschke's expression * for K_n when n is even :

$$K_n = \frac{1}{n![(n-1)!]^{n/2}} ((fa)' \cdots (fa)^n)(f) \quad (n \text{ even}).$$

Theorem. *The mean curvatures $K_{2\nu}$, with even subscript, are represented in (40) as rational integral functions of the coefficients of the first fundamental form and their derivatives.*

§ 7. *Expression of $K_{2\nu+1}$ in terms of the first Fundamental Quantities and Derivatives, when ν is greater than zero.*

Use is made of the determinant theorem

$$(41) \quad \Delta_{k_1 \dots k_{2\nu+1}}^2 = \frac{1}{2} \sum_{j,r}^{1, \dots, n} \begin{vmatrix} \alpha_{ij,k_s} & \alpha_{ij,k_t} \\ \alpha_{ir,k_s} & \alpha_{ir,k_t} \end{vmatrix} \begin{vmatrix} D_{ij,k_s} & D_{ij,k_t} \\ D_{ir,k_s} & D_{ir,k_t} \end{vmatrix} \quad (s, t=1, \dots, 2\nu+1; s \neq t),$$

where $\nu \neq 0$ and the D 's are cofactors of the corresponding α 's in $\Delta_{k_1 \dots k_{2\nu+1}}^2$ and are therefore all of even order and expressible by (39). The results are

$$\begin{aligned} D_{ij,k_s} &= \frac{\epsilon^\nu}{(2\nu)!} F'_{ij} (FA)'_{k_s}, & D_{ij,k_t} &= \frac{\epsilon^\nu}{(2\nu)!} \Phi'_{ij} (\Phi B)'_{k_t}, \\ D_{ir,k_s} &= \frac{\epsilon^\nu}{(2\nu)!} F'_{ir} (FA)'_{k_s}, & D_{ir,k_t} &= \frac{\epsilon^\nu}{(2\nu)!} \Phi'_{ir} (\Phi B)'_{k_t}, \end{aligned}$$

where F'_{ij} is the cofactor of f'_{ij} in $\{f'_{i_1} \cdots f'_{i_{2\nu+1}}\}$, \dots , $(\Phi B)'_{k_t}$ is the cofactor of $(\phi b)'_{k_t}$ in $\{(\phi b)'_{k_1} \cdots (\phi b)'_{k_{2\nu+1}}\}$. Also, by M. I. (120),

$$\begin{vmatrix} \alpha_{ij,k_s} & \alpha_{ij,k_t} \\ \alpha_{ir,k_s} & \alpha_{ir,k_t} \end{vmatrix} = \epsilon (fc)'_{k_s} (\phi c)'_{k_t} \begin{vmatrix} f'_{ij} & \phi'_{ij} \\ f'_{ir} & \phi'_{ir} \end{vmatrix},$$

Substituting in (41), we find

$$\begin{aligned} \Delta_{k_1 \dots k_{2\nu+1}}^2 &= \frac{\epsilon^{2\nu+1}}{[(2\nu)!]^2} (fc)'_{k_s} (FA)'_{k_s} (\phi c)'_{k_t} (\Phi B)'_{k_t} \\ &\quad \times \frac{1}{2} \sum_{j,r}^{1, \dots, 2\nu+1} \begin{vmatrix} f'_{ij} & \phi'_{ij} \\ f'_{ir} & \phi'_{ir} \end{vmatrix} \begin{vmatrix} F'_{ij} & \Phi'_{ij} \\ F'_{ir} & \Phi'_{ir} \end{vmatrix}. \end{aligned}$$

This last sum expands into

$$\begin{aligned} \frac{1}{2} \sum_{j,r}^{1, \dots, 2\nu+1} & [f'_{ij} F'_{ij} \phi'_{ir} \Phi'_{ir} - f'_{ij} \Phi'_{ij} \phi'_{ir} F'_{ir} - f'_{ir} \Phi'_{ir} \phi'_{ij} F'_{ij} + f'_{ir} F'_{ir} \phi'_{ij} \Phi'_{ij}] \\ &= \begin{vmatrix} \{f'_{i_1} \cdots f'_{i_{2\nu+1}}\} & \{f'_{i_1} \phi'_{i_2} \cdots \phi'_{i_{2\nu+1}}\} \\ \{\phi'_{i_1} f'_{i_2} \cdots f'_{i_{2\nu+1}}\} & \{\phi'_{i_1} \cdots \phi'_{i_{2\nu+1}}\} \end{vmatrix}, \end{aligned}$$

so that

$$\begin{aligned} \Delta_{k_1 \dots k_{2\nu+1}}^2 &= \frac{\epsilon^{2\nu+1}}{[(2\nu)!]^2} (fc)'_{k_s} (FA)'_{k_s} (\phi c)'_{k_t} (\Phi B)'_{k_t} \\ &\quad \times \begin{vmatrix} \{f'_{i_1} \cdots f'_{i_{2\nu+1}}\} & \{f'_{i_1} \phi'_{i_2} \cdots \phi'_{i_{2\nu+1}}\} \\ \{\phi'_{i_1} f'_{i_2} \cdots f'_{i_{2\nu+1}}\} & \{\phi'_{i_1} \cdots \phi'_{i_{2\nu+1}}\} \end{vmatrix}. \end{aligned}$$

* K.-G. C., (29).

By (41) this equation holds for all values of s and t from 1 to $2\nu + 1$ except $s = t$. When $s = t$, the second member vanishes. Sum the equations given by using all values of s and t from 1 to $2\nu + 1$ and divide by $(2\nu + 1)2\nu$; also multiply by β^4 . Then

$$(42) \quad \beta^4 \Delta_{i_1 \dots i_{2\nu+1}}^2 = \frac{\varepsilon^{2\nu+1}}{(2\nu+1)(2\nu)[(2\nu)!]^2} ((fc)'_{k_1} (fa)_{k_2}^2 \dots (fa)_{k_{2\nu+1}}^{2\nu+1}) \\ \times ((\phi c)'_{k_1} (\phi b)_{k_2}^2 \dots (\phi b)_{k_{2\nu+1}}^{2\nu+1}) \left| \begin{array}{cc} \{f'_{i_1} \dots f_{i_{2\nu+1}}^{2\nu+1}\} & \{f'_{i_1} \phi_{i_2}^2 \dots \phi_{i_{2\nu+1}}^{2\nu+1}\} \\ \{\phi'_{i_1} f_{i_2}^2 \dots f_{i_{2\nu+1}}^{2\nu+1}\} & \{\phi'_{i_1} \dots \phi_{i_{2\nu+1}}^{2\nu+1}\} \end{array} \right|.$$

And by (34)

$$K_{2\nu+1} = \sum_{i_1 \dots i_{2\nu+1} k_1 \dots k_{2\nu+1}}^{1, \dots, n} [\beta^4 \Delta_{i_1 \dots i_{2\nu+1}}^2]_{k_1 \dots k_{2\nu+1}}^{\frac{1}{2}} A_{i_1 \dots i_{2\nu+1}} \quad (\nu > 0).$$

Thus by (34) and (42) we have $K_{2\nu+1} (\nu > 0)$ expressed in terms of the first fundamental quantities and derivatives (but only in the irrational form of a sum of square roots).

The case of K_1 presents special difficulty:

$$K_1 = \beta^2 \sum_{ik}^{1, \dots, n} \alpha_{ik} A_k^i.$$

In K.-G. C. (p. 24), Maschke suggests a method for expressing the α 's in terms of the a 's when n is odd. His formula (24) should, however, be written,

$$(43) \quad \alpha_{11} \Delta^{n-2} = \begin{vmatrix} A_{22} & \dots & A_{2n} \\ A_{n2} & \dots & A_{nn} \end{vmatrix}.$$

If n is odd, the elements of the second member of (43) are of even order, and therefore expressible by (39), and similarly for every α . But Δ itself is of odd order, and is raised to an odd power ($n - 2$ instead of $n - 1$).^{*} Equation (43) is true also for even values of n , so that the α 's are always expressible by (43) in terms of the first fundamental quantities and derivatives (if $n > 2$), but in all cases irrationally.

Using (43), the author has calculated irrational values of K_1 when n is greater than two; but the notation is so complicated that the presentation of the results seems impracticable, if not also useless.[†]

If $2\nu + 1 = n$, the sum reduces to a single term and formulas (34) and (42)

^{*} Cf. BÔCHER, *Introduction to Higher Algebra*, § 11.

[†] In a recent paper the author has calculated the value of K_1 as well as of the other curvatures of odd subscript, for a space of $n - 1$ dimensions defined in R_n by the equation $U(x_1 \dots x_n) = 0$. These values involve only the coefficients of the first fundamental form of R_n and their derivatives, together with the function U .

give a rational value for K_n^2 ,

$$(44) \quad K_n^2 = \beta^4 \Delta^2 = \frac{\epsilon^{n+2}}{n(n-1)} \left((fc)'(fa)^2 \cdots (fa)^n \right) \left((\phi c)'(\phi b)^2 \cdots (\phi b)^n \right) \begin{vmatrix} (f) & (f'\phi) \\ (\phi'f) & (\phi) \end{vmatrix}.$$

By the method used in K.-G. C. (p. 86), this may be reduced to Maschke's form (31):*

$$(45) \quad K_n^2 = \frac{1}{n[(n-1)!]^{n+2}} \left((fc)'(fa)^2 \cdots (fa)^n \right) \left((\phi c)'(\phi b)^2 \cdots (\phi b)^n \right) (f'\phi'f)(f^2\phi^2\phi).$$

The rather unsatisfactory results of this section are then as follows:

If n is odd, K_n^2 is expressed by (45) as a rational function of the first fundamental quantities and their derivatives. Equations (34) and (42) give irrational expressions for the curvatures of odd index except K_1 , for which no expression is here given.

PART II.

INVARIANTS OF R_λ IN R_n .

The quantities $K_{2\nu}$ and K_n^2 , for n odd, are by their forms (40) and (45) differential invariants of the first fundamental quadratic form (3). When (3) defines the arc-element of a space R_n of n dimensions contained in an euclidean space S_{n+1} of $n+1$ dimensions, these K 's have the geometric meaning already assigned to them. It is our object † to find corresponding invariants of a space R_λ of λ dimensions, represented as differential parameters of a general space R_n of higher dimensions containing R_λ .

§ 1. Definitions and Preliminary Formulas.

In the general space R_n , of n dimensions, whose coördinates are x_1, \dots, x_n and whose arc-element is defined by equation (3), let the space R_λ of λ dimensions ($\lambda < n$) be defined by the $n - \lambda$ equations

$$(46) \quad U^{\lambda+1}(x_1, \dots, x_n) = \text{const.}, \dots, U^n(x_1, \dots, x_n) = \text{const.}$$

If λ other arbitrarily chosen functions of x_1, \dots, x_n , say u', \dots, u^λ , such that

$$\Delta = (u' \dots u^\lambda U^{\lambda+1} \dots U^n) \neq 0,$$

are adjoined to these, the space R_λ may also be represented in parametric form

$$(47) \quad x_1 = x_1(u', \dots, u^\lambda), \dots, x_n = x_n(u', \dots, u^\lambda),$$

* In MASCHKE's reduction there are two slight numerical errors which balance each other. His equation (30) differs from (44) above in that he has divided by n^2 instead of by $n(n-1)$; while in his reduction of (30) there are $n-1$ of the terms which become equal, instead of n .

† Cf. K.-G. C., § 5.

by solving the $n - \lambda$ equations (46) with the λ equations

$$(48) \quad u'(x_1, \dots, x_n) = u', \dots, u^\lambda(x_1, \dots, x_n) = u^\lambda.$$

Any n differentials satisfying the $n - \lambda$ equations, found by differentiating (46),

$$\sum_{i=1}^n U_i^{\lambda+1} dx_i = 0, \dots, \sum_{i=1}^n U_i^n dx_i = 0$$

determine a certain direction in R_λ . In order to find these differentials in terms of du', \dots, du^λ , we differentiate also equations (48) and solve the set

$$\begin{aligned} u'_1 dx_1 + \dots + u'_n dx_n &= du', \\ \cdot &\cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ u^\lambda_1 dx_1 + \dots + u^\lambda_n dx_n &= du^\lambda, \\ U_1^{\lambda+1} dx_1 + \dots + U_n^{\lambda+1} dx_n &= 0, \\ \cdot &\cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ U_1^n dx_1 + \dots + U_n^n dx_n &= 0. \end{aligned}$$

If A^{kr} be the cofactor of u_r^k in Δ , then

$$dx_r = \frac{1}{\Delta} \sum_{k=1}^\lambda A^{kr} du^k$$

and therefore,

$$(49) \quad \sum_{r=1}^n p_r dx_r = \frac{1}{\Delta} \sum_{k=1}^\lambda \{u' \dots u^{k-1} p u^{k+1} \dots u^\lambda U\} du^k,$$

where p is any ordinary function of x_1, \dots, x_n .

In order to find the expression for ds in terms of u', \dots, u^λ , we introduce for the differential quantic (3) the symbolic form

$$ds^2 = \sum_{i,k}^{1, \dots, n} a_{ik} dx_i dx_k = \left[\sum_{i=1}^n f_i dx_i \right]^2.$$

Then (49) gives for the length element in R_λ

$$\begin{aligned} (50) \quad ds^2 &= \frac{1}{\Delta^2} \left[\sum_{i=1}^\lambda \{u' \dots u^{i-1} f u^{i+1} \dots u^\lambda U\} du_i \right]^2 \\ &= \frac{1}{\beta^2 \Delta^2} \left[\sum_{i=1}^\lambda (u' \dots u^{i-1} f u^{i+1} \dots u^\lambda U) du_i \right]^2. \end{aligned}$$

We may also introduce for ds^2 , as given in terms of u', \dots, u^λ , the symbolic form

$$(51) \quad ds^2 = \left[\sum_{i=1}^\lambda f_i du^i \right]^2.$$

By comparing (50) and (51) we find

$$(52) \quad f_i = \frac{1}{\Delta} \{u' \dots u^{i-1} f u^{i+1} \dots u^\lambda U\} = \frac{1}{\beta \Delta} (u' \dots u^{i-1} f u^{i+1} \dots u^\lambda U).$$

If we use the symbols of form (51), the invariants $K_{2\nu}$ and K_λ^2 (λ odd) of R_λ may be written, by (40) and (45),

$$(53) \quad (2\nu)!(\lambda - 2\nu)![(\lambda - 1)!]^\nu K_{2\nu} = G_{2\nu} = ((fa)' \cdots (fa)^{2\nu} \bar{f}^{2\nu+1} \cdots \bar{f}^\lambda)(\bar{f}' \cdots \bar{f}^\lambda),$$

$$(54) \quad \lambda [(\lambda - 1)!]^{\lambda+2} K_\lambda^2 = G_\lambda^2 = ((fc)'(fa)^2 \cdots (fa)^\lambda) \\ \times ((gc)'(gb)^2 \cdots (gb)^\lambda)(\bar{f}'g'\bar{f}^3 \cdots \bar{f}^\lambda)(\bar{f}^2g^2 \cdots g^\lambda),$$

where $G_{2\nu}$ and G_λ^2 are introduced merely for convenience. In all invariantive brackets containing the new symbols, of the quadratic form (51), the differentiation is with respect to the λ variables u', \dots, u^λ . This is indicated sufficiently by the German type and the number of symbols inside the brackets. β_u is defined by the equation

$$(\bar{f}' \cdots \bar{f}^\lambda) = \beta_u \{ \bar{f}' \cdots \bar{f}^\lambda \}.$$

We now proceed to compute the values of the invariantive expressions used in (53) and (54) in terms of the symbols of the first fundamental form (3), of R_n and the functions $U^{\lambda+1}, \dots, U^n$ which define R_λ in R_n .

By means of (52) and D. P. (3), we obtain

$$\{ \bar{f}' \cdots \bar{f}^\lambda \} = \frac{1}{\Delta^\lambda} \{ f' \cdots f^\lambda U \} \{ u' \cdots u^\lambda U \}^{\lambda-1} = \frac{1}{\Delta} \{ f' \cdots f^\lambda U \},$$

so that

$$(55) \quad \frac{1}{\beta_u} (\bar{f}' \cdots \bar{f}^\lambda) = \frac{1}{\beta \Delta} (f' \cdots f^\lambda U).$$

To calculate the value of β_u , square (55) and simplify the result by placing $(\bar{f}' \cdots \bar{f}^\lambda)^2 = \lambda!$, according to M. I. (17), and $(f' \cdots f^\lambda U)^2 = \lambda! (n - \lambda)! \Delta^{n-\lambda} U$ by (38). This gives

$$(56) \quad \beta_u = \omega \beta \Delta, \quad \omega = \sqrt{\frac{1}{(n - \lambda)! \Delta^{n-\lambda} U}}.$$

Then

$$(\bar{f}' \cdots \bar{f}^\lambda) = \omega (f' \cdots f^\lambda U).$$

The other invariantive forms in (53) and (54) are reduced by the same method, and by interchanging equivalent symbols, giving*

$$(57) \quad (\bar{f}'g'\bar{f}^3 \cdots \bar{f}^\lambda) = \omega (f'g'f^3 \cdots f^\lambda U), \quad (\bar{f}^2g^2 \cdots g^\lambda) = \omega (f^2g^2 \cdots g^\lambda U) \\ ((fa)' \cdots (fa)^{2\nu} \bar{f}^{2\nu+1} \cdots \bar{f}^\lambda) \\ = \omega (\omega (faU)', \omega (faU)^2, \dots, \omega (faU)^{2\nu}, f^{2\nu+1} \cdots f^\lambda U),$$

*Inside the invariantive brackets, we have followed MASCHKE's custom of omitting commas between symbols, except where ambiguity might occur. Cf. M. I., p. 448.

By applying D. P. (1) to the first two brackets, and proceeding as above, one finds $T=0$ also for even values of k .

With the help of these results (58) becomes

$$G_{2\nu} = \omega^{2\nu+2} ((faU)' \dots (faU)^{2\nu} f^{2\nu+1} \dots f^\lambda U) (f' \dots f^\lambda U).$$

Then, by (53) and (56),

$$(59) \quad K_{2\nu} = \frac{(\lambda-1)! ((faU)' \dots (faU)^{2\nu} f^{2\nu+1} \dots f^\lambda U) (f' \dots f^\lambda U)}{(2\nu)! (\lambda-2\nu)! [(\lambda-1)! (n-\lambda)! \Delta^{n-\lambda} U]^{\nu+1}}.$$

If $2\nu = \lambda$, (59) becomes

$$(60) \quad K_\lambda = \frac{((faU)' \dots (faU)^\lambda U) (f' \dots f^\lambda U)}{\lambda [(\lambda-1)! (n-\lambda)! \Delta^{n-\lambda} U]^{(\lambda+2)/2}},$$

which agrees with Maschke's form, K.-G. C. (60). The symbols f and a belong to the quadratic form (3), expressing the length element of R_n . Further, $(faU)^i = (f^i a^2 \dots a^\lambda U^{\lambda+1} \dots U^n)$, in which f^i is equal to f^i in $(f' \dots f^\lambda U)$, while the sets of symbols $a^2 \dots a^\lambda$ are equal in any two consecutive brackets $(faU)^{2k-1}$, $(faU)^{2k}$ and otherwise distinct.

The result is then that $K_{2\nu}$, for the space R_λ , is expressible rationally in terms of the coefficients of the first fundamental form of R_n and their derivatives, together with the functions $U^{\lambda+1}, \dots, U^n$ (which define R_λ in R_n) and their derivatives.

§ 3. Expression for K_λ^2 when λ is odd.*

The invariant K_λ^2 (λ odd) can be expressed in a manner similar to the above. Substituting from (57) into (54), one gets

$$(61) \quad G_\lambda^2 = \omega^4 (\omega(fcU)', \omega(faU)^2, \dots, \omega(faU)^\lambda U) \\ \times (\omega(\phi cU)', \omega(\phi bU)^2, \dots, \omega(\phi bU)^\lambda U) (f' \phi' f^3 \dots f^\lambda U) (f^2 \phi^2 \dots \phi^\lambda U).$$

By D. P. (4),

$$\begin{aligned} (\omega(fcU)', \omega(faU)^2, \dots, \omega(faU)^\lambda U) &= \omega^\lambda ((fcU)' (faU)^2 \dots (faU)^\lambda U) \\ &+ \omega^{\lambda-1} (fcU)' (\omega, (faU)^2 \dots (faU)^\lambda U) \\ &+ \omega^{\lambda-1} \sum_{i=2}^{\lambda} (faU)^i ((fcU)' (faU)^2 \dots (faU)^{i-1}, \omega, (faU)^{i+1} \dots (faU)^\lambda U) \\ &\equiv \omega^\lambda \alpha_1 + \omega^{\lambda-1} \alpha_2 + \omega^{\lambda-1} \alpha_3. \\ (\omega(\phi cU)', \omega(\phi bU)^2, \dots, \omega(\phi bU)^\lambda U) &= \omega^\lambda ((\phi cU)' (\phi bU)^2 \dots (\phi bU)^\lambda U) \\ &+ \omega^{\lambda-1} (\phi cU)' (\omega, (\phi bU)^2 \dots (\phi bU)^\lambda U) \\ &+ \omega^{\lambda-1} \sum_{k=2}^{\lambda} (\phi bU)^k ((\phi cU)' (\phi bU)^2 \dots (\phi bU)^{k-1}, \omega, (\phi bU)^{k+1} \dots (\phi bU)^\lambda U) \\ &\equiv \omega^\lambda \beta_1 + \omega^{\lambda-1} \beta_2 + \omega^{\lambda-1} \beta_3. \end{aligned}$$

* See K.-G. C., p. 93.

$$\alpha, \beta, \gamma \delta = (1 - \lambda) \alpha, \beta, \gamma \delta.$$

