# AN APPLICATION OF SYMBOLIC METHODS TO THE TREATMENT OF MEAN CURVATURES IN HYPERSPACE\*

BY

#### WILLIAM HUNT BATES

This paper is an application of MASCHKE's symbolic method for discussing invariants of quadratic differential forms, as developed in his article, A Symbolic Treatment of the Theory of Invariants of Quadratic Differential Quantics of n Variables.† Extensive use is also made of results and methods contained in two later publications, Differential Parameters of the First Order,‡ and The Kronecker-Gaussian Curvature of Hyperspace. § Some familiarity with these three articles is implied.

Part I of the present paper is devoted to the study of the curvatures of an n-space  $R_n$  in an euclidean (n+1)-space  $S_{n+1}$ . In §§ 1-3 the equations and some of the properties of the lines of curvature of  $R_n$  in  $S_{n+1}$  are developed. In particular, equation (28) gives the n curvatures of the n lines of curvature through a given point of  $R_n$ . The coefficients  $K_1, \dots, K_n$  of this equation are the so-called curvatures of  $R_n$  in  $S_{n+1}$ , involving the coefficients  $a_{ik}$  and  $a_{ik}$  of the two fundamental forms of  $R_n$ . With the help of his symbolic method,  $\|$  MASCHKE has expressed  $K_n$ , when n is even, and  $K_n^2$ , when n is odd, as rational integral functions of the coefficients  $a_{ik}$  of the first fundamental form and their derivatives.

In §§ 4-6 similar expressions are derived for all the curvatures  $K_{2\nu}$  of even index. It does not seem possible to obtain rational results for the curvatures  $K_{2\nu+1}$  of odd index. In § 7, however, it is shown that, with the exception of  $K_1$ , these curvatures are expressible irrationally in terms of the first fundamental quantities and their derivatives.

The symbolic expressions for  $K_{2\nu}$  and  $K_n^2$  show at once that they are differential invariants of the first fundamental quadratic form for  $R_n$ , and they have meaning as invariants of any quadratic form in n variables. Part II of this paper considers a space  $R_{\lambda}$  defined in a space  $R_n(n > \lambda)$ , which is not neces-

<sup>\*</sup> Presented to the Society December 31, 1910.

<sup>†</sup>These Transactions, vol. 4 (1903), pp. 445-469. This paper is referred to hereafter as M. I.

<sup>‡</sup> Ibid., vol. 7 (1906), pp. 69-80; referred to as D. P.

<sup>¿</sup> Ibid., vol. 7 (1906), pp. 81-93; referred to as K.-G. C.

<sup>||</sup> In K.-G. C.

sarily euclidean. The invariants  $K_{2\nu}$  and  $K_{\lambda}^2$  for  $R_{\lambda}$  are calculated in terms of the coefficients  $a_{ik}$  belonging to the length element of  $R_n$  and of the functions  $U^{\lambda+1}, \dots, U^n$  which determine the space  $R_{\lambda}$  in  $R_n$ .

### PART I.

CURVATURES OF AN n-SPACE IN AN (n+1)-SPACE.

§ 1. Parametric Representation for an n-space in an (n+1)-space.

Let  $z', z^2, \dots, z^{n+1}$  be the coördinates \* of an euclidean space  $S_{n+1}$  of n+1 dimensions, i. e., a space whose arc-element is of the form

(1) 
$$ds^{2} = \sum_{i=1}^{n+1} [dz^{i}]^{2}.$$

We define in  $S_{n+1}$  any hypersurface, or space  $R_n$ , of n dimensions, by expressing each z as a function of n independent variables  $x_1, \dots, x_n$ :

The arc-element of  $R_n$  is given by the equation

(3) 
$$ds^2 = \sum_{i,k}^{1,\dots,n} a_{ik} dx_i dx_k,$$

where

(4) 
$$a_{ik} = \sum_{j=1}^{n+1} \frac{\partial z^j}{\partial x_i} \frac{\partial z^j}{\partial x_k} = \sum_{j=1}^{n+1} z_i^j z_k^j = f_i f_k,$$

differentiation with respect to  $x_i$  being indicated here, as in the following pages, by the lower index i.

A space of  $\lambda$  dimensions,  $\lambda < n$ , would be obtained by using in (2) only  $\lambda$  independent variables  $x_1, \dots, x_{\lambda}$ . In particular, a curve in  $S_{n+1}$  is obtained by expressing each z as a function of one new variable x.

If in (2) one puts  $x_2 = \cdots = x_n = 0$ , the resulting curve is called the  $x_1$ -axis of a curvilinear system of coördinates on  $R_n$ . By letting  $x_2, \dots, x_n$  represent arbitrary constants, one gets the complete system of  $x_1$ -curves; and similarly for the other cases.

(5) 
$$x_1\text{-curves if } x_1 \text{ varies and } x_2, \dots, x_n \text{ are constants,} \\ x_n\text{-curves if } x_n \text{ varies and } x_1, \dots, x_{n-1} \text{ are constants.}$$

Equations (3) and (5) give the elements of the new axes,

(6) 
$$ds_1^2 = a_{11} dx_1^2, \cdots, ds_n^2 = a_{nn} dx_n^2,$$

<sup>\*</sup>It is assumed that no confusion will arise from writing the upper index, as MASCHKE does, without parentheses. Exponents are only occasionally used, and will be easily recognized.

where  $ds_k$  is the element of the  $x_k$ -axis. Represent the direction cosines of the  $x_k$ -axis, in the old system, by  $\cos(z', x_k), \dots, \cos(z^{n+1}, x_k)$ . Then

(7) 
$$\cos(z^{j}, x_{k}) = \frac{dz^{j}}{ds_{k}} = \frac{z_{k}^{j} dx_{k}}{\sqrt{a_{kk}} dx_{k}} = \frac{z_{k}^{j}}{\sqrt{a_{kk}}} \quad (j=1, \dots, n+1; k=1, \dots, n).$$

Let  $\omega_{ik}$  be the angle between the  $x_i$ -axis and the  $x_k$ -axis. Then

(8) 
$$\cos \omega_{ik} = \sum_{j=1}^{n+1} \cos (z^j x_i) \cos (z^j x_k) = \sum_{j=1}^{n+1} \frac{z_i^j z_k^j}{\sqrt{a_{ii}} \sqrt{a_{ik}}} = \frac{a_{ik}}{\sqrt{a_{ii}} \sqrt{a_{ik}}} (i, k=1, \dots, n),$$

so that necessary and sufficient conditions for mutual orthogonality of the axes of the new system are

(9) 
$$a_{ik} = 0$$
  $(i, k = 1, \dots, n; i + k).$ 

§ 2. General Curves on  $R_{x}$ .

A general curve on  $R_n$  may be defined by means of n-1 equations,

(10) 
$$U^2(x_1, \dots, x_n) = \text{const.}, \dots, U^n(x_1, \dots, x_n) = \text{const.}$$

The differential equations of this curve, which we call the U-curve, are

(11) 
$$\sum_{i=1}^{n} U_{i}^{2} dx_{i} = 0, \dots, \sum_{i=1}^{n} U_{i}^{n} dx_{i} = 0.$$

Its direction is defined by the ratios of  $dx_1, \dots, dx_n$  in (11). In order to solve for these differentials, let p be any function of  $x_1, \dots, x_n$  which satisfies the condition \*

$$D = (pU^2 \cdots U^n) = (pU) \neq 0.$$

If  $A^r$  denotes the cofactor of  $p_r$  in D, equations (11) are identically satisfied by

(12) 
$$dx_1 = \rho A', \cdots, dx_n = \rho A^n,$$

where  $\rho$  is an arbitrary parameter.

The direction cosines  $\xi'$ , ...,  $\xi^{n+1}$  of the *U*-curve are found as follows. From (12),

(13) 
$$\sum_{i=1}^{n} p_{i} dx_{i} = \rho \sum_{i=1}^{n} p_{i} A^{i} = \rho (p U).$$

Then

$$\xi^{k} = \frac{dz^{k}}{ds} = \frac{1}{ds} \sum_{i=1}^{n} z_{i}^{k} dx_{i} = \frac{\rho}{ds} (z^{k} U),$$

where ds is arc-element of the U-curve. Now

$$\sum_{k=1}^n \left[ \, \xi^k \, \right]^2 = 1 = \sum_{k=1}^{n+1} \left[ \, \frac{\rho}{ds} \, \right]^2 (z^k U)^2 = \left[ \, \frac{\rho}{ds} \, \right]^2 (f U)^2.$$

<sup>\*</sup>See M. I., § 2, for an explanation of this invariantive notation.

Hence

$$\frac{\rho}{ds} = \frac{1}{\sqrt{(fU)^2}}.$$

Then the direction cosines of the U-curve on  $R_n$ , referred to the original system of axes, are

(14) 
$$\xi' = \frac{(z'U)}{\sqrt{(fU)^2}}, \quad \cdots, \quad \xi^{n+1} = \frac{(z^{n+1}U)}{\sqrt{(fU)^2}}.$$

If there is given also a V-curve on  $R_n$  by equations similar to (10), its direction cosines may be written

(15) 
$$\eta' = \frac{(z'V)}{\sqrt{(fV)^2}}, \quad \dots, \quad \eta^{n+1} = \frac{(z^{n+1}V)}{\sqrt{(fV)^2}}.$$

If  $\omega$  is the angle between the two curves, we have from (14) and (15)

(16) 
$$\cos \omega = \sum_{i=1}^{n+1} \xi^i \eta^i = \sum_{i=1}^{n+1} \frac{(z^i U)(z^i V)}{\sqrt{(f U)^2} \sqrt{(f V)^2}} = \frac{(f U)(f V)}{\sqrt{(f U)^2} \sqrt{(f V)^2}}.$$

Thus a necessary and sufficient condition for orthogonality of the two curves is

$$(17) (fU)(fV) = 0.$$

Equation (17) also defines the orthogonal trajectories of a system of U-curves on  $R_n$ . An illustration is found in the case of curves on an ordinary surface.

A line L drawn on  $R_n$  such that the normals to  $R_n$  along L (with respect to the enclosing space  $S_{n+1}$ ) generate a developable surface is called \* a line of curvature of  $R_n$  in  $S_{n+1}$ .

At a point P of  $R_n$  there is a unique normal to  $R_n$  in  $S_{n+1}$ . Let the direction cosines of this normal be  $\zeta'$ ,  $\cdots$ ,  $\zeta^{n+1}$ . Choose P as origin of the system of x-axes on  $R_n$ . Then, since the normal to  $R_n$  at P is orthogonal to every direction on  $R_n$  at P, we have from (7)

(18) 
$$\sum_{i=1}^{n+1} \zeta^i z_k^i = 0 \qquad (k=1,\dots,n).$$

The coefficients  $a_{ik}$  of the first fundamental form of  $R_n$ , given in (3) are the first fundamental quantities. The second fundamental quantities are defined by the equations

$$\alpha_{ik} = \sum_{j=1}^{n+1} \zeta^{j} z_{ik}^{j} \qquad (i, k=1, \dots, n).$$

<sup>\*</sup> Cf. BIANCHI, Lezioni di Geometria Differenziale, vol. I, p. 125.

By differentiating (18), one obtains

(19) 
$$\alpha_{ik} = \sum_{j=1}^{n+1} \zeta^j z_{ik}^j = -\sum_{j=1}^{n+1} \zeta_i^j z_k^j \qquad (i, k=1, \dots, n).$$

The letters q and  $\gamma$  are to be used in this paper as symbols of the second fundamental quantities

(20) 
$$\alpha_{ik} = g_i g_k = \gamma_i \gamma_k = \alpha_{ki}.$$

Let C be the curve of  $S_{n+1}$  which is the envelope of the normals along L. Let  $M(z', \dots, z^{n+1})$  be any point of L, and  $\overline{M}(\overline{z}', \dots, \overline{z}^{n+1})$  be the point where the normal at M meets C. Denote by r the distance  $M\overline{M}$ , which is positive or negative according to the direction of  $\overline{M}$  from M.

(21) 
$$\overline{z}' = z' - r\zeta', \dots, \overline{z}^{n+1} = z^{n+1} - r\zeta^{n+1}$$

Take derivatives of equations (21) with respect to the arc s of L. Then, since C is envelope of the normals along L,

(22) 
$$\frac{d\overline{z}'}{ds} = \frac{dz'}{ds} - r \frac{d\zeta'}{ds} - \zeta' \frac{dr}{ds} = q\zeta',$$

$$\frac{d\overline{z}^{n+1}}{ds} = \frac{dz^{n+1}}{ds} - r \frac{dz^{n+1}}{ds} - \zeta^{n+1} \frac{dr}{ds} = q\zeta^{n+1},$$

where q is a factor of proportionality to be determined. Multiply equations (22) by  $\zeta', \ldots, \zeta^{n+1}$  in order and add. We get

(23) 
$$q \sum_{i=1}^{n+1} [\xi^{i}]^{2} = \sum_{i=1}^{n+1} \zeta^{i} \frac{dz^{i}}{ds} - r \sum_{i=1}^{n+1} \zeta^{i} \frac{d\zeta^{i}}{ds} - \frac{dr}{ds} \sum_{i=1}^{n+1} [\zeta^{i}]^{2}.$$
Now
$$\sum_{i=1}^{n+1} [\xi^{i}]^{2} - 1 \sum_{i=1}^{n+1} \xi^{i} \frac{d\zeta^{i}}{ds} - 0 \sum_{i=1}^{n+1} \xi^{i} \frac{dz^{i}}{ds} - 0$$

$$\sum_{i=1}^{n+1} \left[ \zeta^i \, \right]^2 = 1 \,, \qquad \sum_{i=1}^{n+1} \zeta^i \, rac{d \zeta^i}{ds} = 0 \,, \qquad \sum_{i=1}^{n+1} \zeta^i rac{d z^i}{ds} = 0 \,,$$

since  $\zeta'$ , ...,  $\zeta^{n+1}$  are direction cosines of the normal and dz'/ds, ...,  $dz^{n+1}/ds$ are direction cosines of L. Substituting these results in (23), one obtains

$$q = -\frac{dr}{ds}.$$

Then equations (22) give

(24) 
$$\frac{dz'}{ds} = r \frac{d\zeta'}{ds}, \dots, \frac{dz^{n+1}}{ds} = r \frac{d\zeta^{n+1}}{ds},$$

 $\mathbf{or}$ 

(25) 
$$\frac{dz'}{d\zeta} = \frac{dz^2}{d\zeta^2} = \cdots = \frac{dz^{n+1}}{d\zeta^{n+1}} = r.$$

This result is expressed in curvilinear coördinates as follows:

$$z'_{1} dx_{1} + \cdots + z'_{n} dx_{n} = r(\zeta'_{1} dx_{1} + \cdots + \zeta'_{n} dx_{n}),$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$z_{1}^{n+1} dx_{1} + \cdots + z_{n}^{n+1} dx_{n} = r(\zeta'_{1}^{n+1} dx_{1} + \cdots + \zeta'_{n}^{n+1} dx_{n}).$$

Multiply these equations in order by  $z'_k, \dots, z^{n+1}_k$  and add. We find

$$\sum_{i=1}^{n+1} z_k^i z_1^i dx_1 + \cdots + \sum_{i=1}^{n+1} z_k^i z_n^i dx_n = r \left[ \sum_{i=1}^{n+1} z_k^i \zeta_1^i dx_1 + \cdots + \sum_{i=1}^{n+1} z_k^i \zeta_n^i dx_n \right].$$

Then, by (4) and (19),

(26) 
$$a_{\nu} dx_1 + \cdots + a_{\nu} dx_n = -r \left[ \alpha_{\nu} dx_1 + \cdots + \alpha_{\nu} dx_n \right] \quad (k=1, \dots, n).$$

Equations (26) hold for every line of curvature on  $R_n$ . Conversely, if equations (26) be true for any curve L on  $R_n$ , then there exists a curve C in  $S_{n+1}$  whose tangents are normal to  $R_n$  along L, so that L is a line of curvature on  $R_n$  by definition.

When equations (26) are written in the form

(27) 
$$(a_{11} + \alpha_{11}r)dx_1 + \dots + (a_{1n} + \alpha_{1n}r)dx_n = 0,$$

$$(a_{n1} + \alpha_{n1}r)dx_1 + \dots + (a_{nn} + \alpha_{nn}r)dx_n = 0,$$

it is evident that the curvature (1/r) of each line of curvature through a point P on  $R_n$  must satisfy the condition

(28) 
$$\begin{vmatrix} a_{11} + \alpha_{11}r & a_{12} + \alpha_{12}r & \cdots & a_{1n} + \alpha_{1n}r \\ a_{21} + \alpha_{21}r & a_{22} + \alpha_{22}r & \cdots & a_{2n} + \alpha_{2n}r \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} + \alpha_{n1}r & a_{n2} + \alpha_{n2}r & \cdots & a_{nn} + \alpha_{nn}r \end{vmatrix} = 0,$$

since otherwise equations (27) would have no solution except

$$dx_1 = dx_2 = \cdots = dx_n = 0$$
.

Hence the reciprocals of the roots of (28) are exactly the curvatures of the n lines of curvature through a point P on  $R_n$ .

The coefficients of r in (28) are called the curvatures of  $R_n$  in  $S_{n+1}$  and are discussed in §§ 4–7. Before proceeding to that discussion, we derive an important property of lines of curvature.

Expressing (26) in symbolic notation, one finds

(29) 
$$f_k \sum_{i=1}^n f_i dx_i = -rg_k \sum_{i=1}^n g_i dx$$
 (k=1,...,n)

If now a line of curvature be represented as a U-curve (10), one gets from (13) and (29)

(30) 
$$f_{k}(fU) = -rg_{k}(gU) \qquad (k=1,\dots,n).$$

A symmetrical expression for r is obtained by multiplying equations (30) in order by the cofactors of  $f_1, \dots, f_n$  in (fU) and adding:

(31) 
$$r = -\frac{(fU)^2}{(gU)^2}.$$

If any two lines of curvature through P be given as U and V-curves, and their respective curvatures be denoted by 1/r' and 1/r'', one gets from (30)

$$g_1(gU) = -\frac{1}{r'}f_1(fU), \dots, g_n(gU) = -\frac{1}{r'}f_n(fU),$$

$$g_1(gV) = -\frac{1}{r''}f_1(fV), \dots, g_n(gV) = -\frac{1}{r''}f_n(fV).$$

Multiply the equations of the first line in order by the cofactors of  $f_1, \dots, f_n$  in (fV) and add. Also multiply the equations of the second line in order by the cofactors of  $f_1, \dots, f_n$  in (fU) and add. Then

$$(gU)(gV) = -\frac{1}{r'}(fU)(fV) = -\frac{1}{r''}(fU)(fV),$$

so that either r' = r'' or (fU)(fV) = 0. Hence by (17) we have

**Theorem I.** Any two distinct lines of curvature through an ordinary point P of  $R_n$  are orthogonal to each other.

If the lines of curvature through P be taken as parameter lines, then, by (9),

$$a_{ik} = 0 (i + k).$$

It follows at once from (26) that also

$$\alpha_{ik} = 0 \qquad (i, k=1, \dots, n; i+k).$$

**Theorem II.** If the lines of curvature at an ordinary (not umbilic) point of  $R_n$  be taken as parameter lines, then

$$a_{ik} = 0$$
,  $a_{ik} = 0$   $(i, k = 1, \dots, n; i + k)$ .

§ 4. Definition of the Curvatures of  $R_n$  in  $S_{n+1}$ .

Equation (28) may be written in the form

(32) 
$$H_0 + H_1 r + \dots + H_{n-1} r^{n-1} + H_n r^n = 0$$
, where \*  $H_0 = |a_{ik}| = 1/\beta^2$ ,  $H_n = |a_{ik}|$ ,

<sup>\*</sup> M. I., (9).

while for  $j = 1, \dots, n, H_j$  is the sum of all the determinants obtained from  $|a_{ik}|$  by replacing in all possible ways j columns of  $|a_{ik}|$  by the corresponding columns of  $|a_{ik}|$ . Dividing (32) by  $H_0$  one obtains

$$(33) 1 + K_1 r + \cdots + K_{n-1} r^{n-1} + K_n r^n = 0.$$

The coefficient  $K_n$  (the product of all the curvatures) is the Kronecker-Gaussian curvature of hyperspace. It has been shown to be expressible in terms of the first fundamental quantities and their derivatives (cf. K-G. C.). In this paper the coefficients of (33) are called the n curvatures of  $R_n$  in  $S_{n+1}$ . By definition

$$K_1 = eta^2 \sum_{i,\,k}^{1,\,\ldots,\,n} lpha_{ik} A_k^i, \qquad K_2 = eta^2 \sum_{i_1i_2,\,k_1k_2}^{1,\,\ldots,\,n} \left| rac{lpha_{i_1k_1}}{lpha_{i_2k_1}} lpha_{i_1k_2}}{lpha_{i_2k_1}} 
ight| \cdot A_{k_1k_2}^{i_1i_2} = \sum_{i_1i_2k_1k_2}^{1,\,\ldots,\,n} \Delta_{i_1i_2}_{k_2k_2} A_{k_1k_2}^{i_1i_2}, \; \cdots,$$

(34) 
$$K_{m} = \beta^{2} \sum_{i_{1}...i_{m}k_{1}...k_{m}}^{1,...,n} \Delta_{\substack{i_{1}...i_{m} \\ k_{1}...k_{m}}} A_{\substack{i_{1}...i_{m} \\ k_{1}...k_{m}}}^{i_{1}...i_{m}} \qquad (m=1,\cdots,n),$$

where  $A_k^i$  is the cofactor of  $a_{ik}$  in  $|a_{ik}|$ ,  $A_{k_1k_2}^{i_1i_2}$  is the algebraic complement of

$$\begin{vmatrix} a_{i_1k_1}a_{i_1k_2} \\ a_{i_2k_1}a_{i_2k_2} \end{vmatrix}$$

in  $|a_{ik}|$ , while  $\Delta_{\substack{i_1i_2\\k_1k_2}}$  is the second minor of  $|a_{ik}|$  indicated for  $K_2$  above; and similarly for the A's and  $\Delta$ 's in  $K_m$ . Both sets  $i_1, \dots, i_m$  and  $k_1, \dots, k_m$  are considered as being in ascending numerical order.

§ 5. Invariant Symbolic Forms of 
$$K_1, \dots, K_n$$
.

If  $F_k^i$  be the cofactor of  $f_k^i$  in the functional determinant  $\{f', \dots, f^n\}$ , Maschke \* has shown that

$$A_k^i = \frac{1}{(n-1)!} F_i' F_k'.$$

Thus

$$\begin{split} K_1 &= \beta^2 \sum_{i,\,\,k}^{i,\,\,\ldots,\,\,n} \alpha_{ik} \, A_k^{\,i} = \frac{\beta^2}{(\,n-1\,)\,!} \sum_{i,\,\,k}^{i,\,\,\ldots,\,\,n} g_i g_k \, F'_{\,i} \, F'_{\,k} \\ &= \frac{\beta^2}{(\,n-1\,)\,!} \, \{ \, g f^2 \, \cdots f^n \, \}^2 = \frac{1}{(\,n-1\,)\,!} (\, g f)^2 \, . \end{split}$$

This suggests a method for reducing all the curvatures to convenient invariant forms. Let  $F_{i_1 \dots i_m}^{1 \dots m}$  be the algebraic complement of

$$\begin{vmatrix} f'_{i_1} \cdots f'_{i_m} \\ \vdots \\ f^m_{i_1} \cdots f^m_{i_m} \end{vmatrix}$$

<sup>\*</sup> M. I., p. 450.

in  $\{f' \cdots f^n\}$ . Then the product  $F_{i_1 \cdots i_m}^1 \cdot F_{i_1 \cdots i_m}^1 \cdots m$  may be written

$$\begin{vmatrix} f_1^{m+1} \cdots f_{i_1-1}^{m+1} f_{i_1+1}^{m+1} \cdots f_{i_{m-1}}^{m+1} f_{i_{m+1}}^{m+1} \cdots f_{i_n}^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^n \cdots f_{i_1-1}^n f_{i_1+1}^n \cdots f_{i_{m-1}}^n f_{i_{m+1}}^n \cdots f_{i_n}^n \end{vmatrix} F_{k_1 \cdots k_m}^{1 \cdots m}.$$

If the first determinant of this product be expanded, one finds (n-m)! terms of the form

$$(-1)^{\mu} f_1 \cdots f_{i_1-1} f_{i_1+1} \cdots f_{i_m-1} f_{i_m+1} \cdots f_n \cdot F_{k_1 \dots k_m}^{1 \dots m},$$

where the suppressed upper indices of the first factor are understood to be any permutation of the numbers m+1, ..., n, while  $\mu$  represents the number of inversions in the permutation. Since the equivalent symbols  $f^{m+1}$ , ...,  $f^n$  may be interchanged in all possible ways without altering the value of the term, let them be so interchanged for each term as to reduce the first factor to  $(-1)^{\mu}$  times the principal diagonal term of  $F_{i_1...i_m}^{1...m}$ . This causes an interchange of rows in the second (determinant) factor  $F_{i_1...i_m}^{1...m}$  so that it becomes in each case  $(-1)^{\mu}$  times its original form. Hence the above product becomes

$$(n-m)! f_1^{m+1} \cdots f_{i_1-1}^{i_1+m-1} f_{i_1+1}^{i_1+m} \cdots f_{i_m-1}^{i_m} f_{i_m+1}^{i_m+1} \cdots f_n^{n} \cdot F_{k_1 \dots k_m}^{1 \dots m}.$$

Multiplying each f into the corresponding row of the determinant  $F_{k_1 \dots k_m}^{1 \dots m}$  (which has a form similar to that given above for  $F_{i_1 \dots i_m}^{1 \dots m}$ ), we have

$$F_{i_1 \dots i_m}^{1 \dots m} \cdot F_{k_1 \dots k_m}^{1 \dots m} = (n-m)! A_{k_1 \dots k_m}^{i_1 \dots i_m},$$

 $\mathbf{or}$ 

(35) 
$$A_{k_1 \dots k_m}^{i_1 \dots i_m} = \frac{1}{(n-m)!} F_{i_1 \dots i_m}^{1 \dots m} \cdot F_{k_1 \dots k_m}^{1 \dots m}.$$

Also

$$\Delta_{\stackrel{i_1\ldots i_m}{k_1\ldots k_m}} = \begin{vmatrix} \alpha_{i_1k_1}\cdots\alpha_{i_1k_m}\\ \vdots\\ \alpha_{i_mk_1}\cdots\alpha_{i_mk_m} \end{vmatrix} = \begin{vmatrix} g'_{i_1}g'_{k_1}\cdots g'_{i_1}g'_{k_m}\\ \vdots\\ g^m_{i_m}g^m_{k_1}\cdots g^m_{i_m}g^m_{k_m} \end{vmatrix} = g'_{i_1}\cdots g^m_{i_m}\begin{vmatrix} g'_{k_1}\cdots g'_{k_m}\\ \vdots\\ g^m_{k_1}\cdots g^m_{k_m} \end{vmatrix},$$

(36) 
$$\Delta_{i_{1} \dots i_{m}^{m}} = \frac{1}{m!} \begin{vmatrix} g'_{i_{1}} \dots g'_{i_{m}} \\ \vdots & \vdots \\ g^{m}_{i_{1}} \dots g^{m}_{i_{m}} \end{vmatrix} \cdot \begin{vmatrix} g'_{k_{1}} \dots g'_{k_{m}} \\ \vdots & \vdots \\ g^{m}_{k_{1}} \dots g^{m}_{k_{m}} \end{vmatrix}.$$

Substituting (35) and (36) in (34), one finds, by a well-known theorem of determinants,

(37) 
$$K_m = \frac{1}{m!(n-m)!} (g' \cdots g^m f^{m+1} \cdots f^n)^2 = \frac{1}{m!(n-m)!} (g' \cdots g^m f)^2.$$

In particular,

$$K_1 = \frac{1}{(n-1)!} (gf)^2 = \Delta_1 g,$$

where, by M. I. (22),  $\Delta_1 g$  is the first differential parameter of the first quadratic form (3).

Since the other coefficients are corresponding differential parameters (the number of g's being the same as the subscript of K), it would seem fitting to generalize the notation and set \*

(38) 
$$K_{m} = \frac{1}{m!(n-m)!} (g' \cdots g^{m} f)^{2} = \Delta^{m} g,$$

with the note that  $\Delta' g = \Delta_1 g$ .

§ 6. Expression of  $K_{2\nu}$  in terms of the first Fundamental Quantities and Derivatives.

The generalization of the Gauss equation shows that any second order determinant of the second fundamental quantities is equal to a Riemann quadruple index symbol, which is expressible in terms of the first fundamental quantities and derivatives.† By K.-G. C. (27),

$$\begin{vmatrix} \alpha_{i_1k_1}\alpha_{i_1k_2} \\ \alpha_{i_2k_1}\alpha_{i_2k_2} \end{vmatrix} = (i_1i_2k_1k_2) = \frac{1}{(n-1)!}f_{i_1}'f_{i_2}^2 \begin{vmatrix} (fa)_{k_1}'(fa)_{k_2}' \\ (fa)_{k_1}^2(fa)_{k_2}^2 \end{vmatrix}.$$

By an easy induction, any even order determinant of the  $\alpha$ 's is expressed in terms of the symbols of the  $\alpha$ 's as follows:  $\ddagger$ 

(39) 
$$\Delta_{i_{1} \dots i_{2\nu} \atop k_{1} \dots k_{2\nu}} = \frac{\epsilon^{\nu}}{(2\nu)!} \begin{vmatrix} (fa)'_{i_{1}} \cdots (fa)'_{i_{2\nu}} \\ \vdots & \ddots & \vdots \\ (fa)^{2\nu}_{i_{1}} \cdots (fa)^{2\nu}_{i_{2\nu}} \end{vmatrix} \cdot \begin{vmatrix} f'_{k_{1}} \cdots f'_{k_{2\nu}} \\ \vdots & \ddots & \vdots \\ f^{2\nu}_{k_{1}} \cdots f^{2\nu}_{k_{2\nu}} \end{vmatrix},$$

where  $\epsilon = 1/(n-1)!$ ; the symbol  $(fa)^j$  contains  $f^j$  and  $a^2 \cdots a^n$ , while the symbols a in every consecutive pair  $(fa^2 \cdots a^n)^{2\lambda-1}$ ,  $(fa^2 \cdots a^n)^{2\lambda}$  are equal when they have the same index, otherwise they are distinct but equivalent symbols of the first fundamental form (3).

Now from (34), (35), and (39),

$$K_{2\nu} = \frac{\beta^2 \epsilon^{\nu}}{(2\nu)! (n-2\nu)!} \sum_{i_1 \dots i_{2\nu} k_1 \dots k_{2\nu}}^{1 \dots n} \left| \frac{(fa)'_{i_1} \dots (fa)'_{i_{2\nu}}}{\dots \dots \dots \dots} \right| \cdot \left| \frac{(f'_{k_1} \dots f'_{k_{2\nu}})}{\dots \dots \dots \dots} \right| \cdot \left| \frac{(f'_{k_1} \dots f'_{k_{2\nu}})}{(fa)^{2\nu}_{i_1} \dots (fa)^{2\nu}_{i_{2\nu}}} \right| \cdot \left| \frac{(f'_{k_1} \dots f'_{k_{2\nu}})}{(f^{2\nu}_{k_1} \dots f^{2\nu}_{k_{2\nu}})} \right| \times F_{i_1 \dots i_{2\nu}}^{1 \dots 2\nu} F_{k_1 \dots k_{2\nu}}^{1 \dots 2\nu},$$

or

(40) 
$$K_{2\nu} = \frac{\epsilon^{\nu}}{(2\nu)!(n-2\nu)!} ((fa)' \cdots (fa)^{2\nu} f)(f).$$

<sup>\*</sup> The use of  $\Delta_{mg}$  would conflict with the second differential parameter of ordinary differential geometry, which has an entirely different meaning. Cf. BIANCHI, Lezioni di Geometria Differenziale, vol. I, p. 67.

<sup>†</sup>M. I., (117)-(126).

<sup>†</sup> Cf. K.-G. C., (28).

This gives Maschke's expression \* for  $K_n$  when n is even:

$$K_n = \frac{1}{n![(n-1)!]^{n/2}} ((fa)' \cdots (fa)^n)(f) \qquad (n \text{ even}).$$

**Theorem.** The mean curvatures  $K_{2\nu}$ , with even subscript, are represented in (40) as rational integral functions of the coefficients of the first fundamental form and their derivatives.

§ 7. Expression of  $K_{2\nu+1}$  in terms of the first Fundamental Quantities and Derivatives, when  $\nu$  is greater than zero.

Use is made of the determinant theorem

$$(41) \quad \Delta^{2}_{\substack{i_{1} \dots i_{2\nu+1} \\ k_{1} \dots k_{2\nu+1}}} = \frac{1}{2} \sum_{j,\,r}^{1,\,\dots,\,n} \begin{vmatrix} \alpha_{i,k_{s}} & \alpha_{i,k_{t}} \\ \alpha_{i,k_{s}} & \alpha_{i,k_{t}} \end{vmatrix} \cdot \begin{vmatrix} D_{i,k_{s}} & D_{i,k_{t}} \\ D_{i,k_{s}} & D_{i,k_{t}} \end{vmatrix} \quad (s,\,t=1,\,\cdots,\,2\nu+1\,;\,s=t\,),$$

where  $\nu \neq 0$  and the *D*'s are cofactors of the corresponding  $\alpha$ 's in  $\Delta_{k_1 \dots k_{2\nu+1}}^{i_1 \dots i_{2\nu+1}}$  and are therefore all of even order and expressible by (39). The results are

$$\begin{split} D_{i_{j}k_{\bullet}} &= \frac{\epsilon^{\nu}}{(2\nu)!} \, F'_{i_{j}}(\,FA\,)'_{k_{\bullet}}, \qquad D_{i_{j}k_{t}} = \frac{\epsilon^{\nu}}{(2\nu)!} \, \Phi'_{i_{j}}(\,\Phi B\,)'_{k_{t}}, \\ D_{i_{r}k_{\bullet}} &= \frac{\epsilon^{\nu}}{(2\nu)!} \, F'_{i_{r}}(\,FA\,)'_{k_{\bullet}}, \qquad D_{i_{r}k_{t}} = \frac{\epsilon^{\nu}}{(2\nu)!} \, \Phi'_{i_{r}}(\,\Phi B\,)'_{k_{t}}, \end{split}$$

where  $F'_{i_j}$  is the cofactor of  $f'_{i_j}$  in  $\{f'_{i_1} \cdots f^{2\nu+1}_{i_2\nu+1}\}, \cdots, (\Phi B)'_{k_t}$  is the cofactor of  $(\phi b)'_{k_t}$  in  $\{(\phi b)'_{k_1} \cdots (\phi b)^{2\nu+1}_{k_2\nu+1}\}$ . Also, by M. I. (120),

$$\begin{vmatrix} \alpha_{i_jk_*} & \alpha_{i_jk_t} \\ \alpha_{i_rk_*} & \alpha_{i_rk_t} \end{vmatrix} = \epsilon (fc)_{k_*}' (\phi c)_{k_t}' \begin{vmatrix} f_{i_j}' & \phi_{i_j}' \\ f_{i_r}' & \phi_{i_r}' \end{vmatrix},$$

Substituting in (41), we find

$$\begin{split} \Delta^2_{\substack{i_1 \, \dots \, i_{2\nu+1} \\ k_1 \, \dots \, k_{2\nu+1}} &= \frac{\epsilon^{2\nu+1}}{\left[ \, (2\nu) \, ! \, \right]^2} (fc)'_{k_{\bullet}} (FA)'_{k_{\bullet}} (\phi c)'_{k_{\bullet}} (\Phi B)'_{k_{\bullet}} \\ & \times \frac{1}{2} \sum_{j,r}^{1 \, \dots \, 2\nu+1} \left| \begin{matrix} f'_{i_j} & \phi'_{i_j} \\ f'_{i_s} & \phi'_{i_s} \end{matrix} \right| \cdot \left| \begin{matrix} F'_{i_j} & \Phi'_{i_j} \\ F'_{i_s} & \Phi'_{i_s} \end{matrix} \right|. \end{split}$$

This last sum expands into

$$\begin{split} \frac{1}{2} \sum_{j,\,r}^{1,\,\dots,\,2^{\nu+1}} \left[ f_{\,i_{j}}^{\prime} F_{\,i_{j}}^{\prime} \phi_{i_{r}}^{\prime} \Phi_{i_{r}}^{\prime} - f_{\,i_{j}}^{\prime} \phi_{i_{r}}^{\prime} F_{\,i_{r}}^{\prime} - f_{\,i_{r}}^{\prime} \phi_{i_{j}}^{\prime} F_{\,i_{r}}^{\prime} + f_{\,i_{r}}^{\prime} F_{\,i_{r}}^{\prime} \phi_{i_{j}}^{\prime} \Phi_{i_{j}}^{\prime} \right] \\ = \left| \begin{cases} f_{\,i_{1}}^{\prime} \cdots f_{\,i_{2\nu+1}}^{2\nu+1} \\ \{ \phi_{\,i_{1}}^{\prime} f_{\,i_{2}}^{2} \cdots f_{\,i_{2\nu+1}}^{2\nu+1} \end{cases} - \{ f_{\,i_{1}}^{\prime} \phi_{i_{2}}^{2} \cdots \phi_{\,i_{2\nu+1}}^{2\nu+1} \} \right|, \end{split}$$

so that

at
$$\Delta_{i_{1} \dots i_{2\nu+1}}^{2} = \frac{\epsilon^{2\nu+1}}{\left[(2\nu)!\right]^{2}} (fc)'_{k_{\bullet}} (FA)'_{k_{\bullet}} (\phi c)'_{k_{i}} (\Phi B)'_{k_{t}} \\
\times \begin{vmatrix} \{f'_{i_{1}} \dots f^{2\nu+1}_{i_{2\nu+1}}\} & \{f'_{i_{1}} \phi^{2}_{i_{2}} \dots \phi^{2\nu+1}_{i_{2\nu+1}}\} \\
\{\phi'_{i_{1}} f^{2}_{i_{2}} \dots f^{2\nu+1}_{i_{2\nu+1}}\} & \{\phi'_{i_{1}} \dots \phi^{2\nu+1}_{i_{2\nu+1}}\} \end{vmatrix}.$$

<sup>\*</sup> K.-G. C., (29).

By (41) this equation holds for all values of s and t from 1 to  $2\nu + 1$  except s = t. When s = t, the second member vanishes. Sum the equations given by using all values of s and t from 1 to  $2\nu + 1$  and divide by  $(2\nu + 1)2\nu$ ; also multiply by  $\beta^4$ . Then

$$(42) \begin{array}{c} \beta^{4} \Delta_{i_{1} \dots i_{2\nu+1}}^{2} = \frac{\varepsilon^{2\nu+1}}{(2\nu+1)(2\nu)[(2\nu)!]^{2}} \left( (fc)_{i_{1}}^{\prime} (fa)_{k_{2}}^{2} \dots (fa)_{k_{2\nu+1}}^{2\nu+1} \right) \\ \times \left( (\phi c)_{i_{1}}^{\prime} (\phi b)_{k_{2}}^{2} \dots (\phi b)_{k_{2\nu+1}}^{2\nu+1} \right) \left| \begin{cases} f_{i_{1}}^{\prime} \dots f_{i_{2\nu+1}}^{2\nu+1} \\ f_{i_{1}}^{\prime} \dots f_{i_{2\nu+1}}^{2\nu+1} \end{cases} \right. \left. \begin{cases} f_{i_{1}}^{\prime} \phi_{i_{2}}^{2} \dots \phi_{i_{2\nu+1}}^{2\nu+1} \\ f_{i_{2}}^{\prime} \dots f_{i_{2\nu+1}}^{2\nu+1} \end{cases} \right|. \end{array}$$

And by (34)

$$K_{2\nu+1} = \sum_{i_1 \dots i_{2\nu+1} k_1 \dots k_{2\nu+1}}^{1, \dots, n} \left[ \beta^4 \Delta^2_{i_1 \dots i_{2\nu+1}} \right]^{\frac{1}{2}} A_{i_1 \dots i_{2\nu+1} \atop k_1 \dots k_{2\nu+1}} \qquad (\nu > 0).$$

Thus by (34) and (42) we have  $K_{2\nu+1}(\nu>0)$  expressed in terms of the first fundamental quantities and derivatives (but only in the irrational form of a sum of square roots).

The case of  $K_1$  presents special difficulty:

$$K_1 = \beta^2 \sum_{i,k=1}^{1,\ldots,n} \alpha_{ik} A_k^i.$$

In K.-G. C. (p. 24), Maschke suggests a method for expressing the  $\alpha$ 's in terms of the  $\alpha$ 's when n is odd. His formula (24) should, however, be written,

(43) 
$$\alpha_{11} \Delta^{n-2} = \begin{vmatrix} A_{22} \cdots A_{2n} \\ A_{n2} \cdots A_{nn} \end{vmatrix} .$$

If n is odd, the elements of the second member of (43) are of even order, and therefore expressible by (39), and similarly for every  $\alpha$ . But  $\Delta$  itself is of odd order, and is raised to an odd power (n-2) instead of n-1.\* Equation (43) is true also for even values of n, so that the  $\alpha$ 's are always expressible by (43) in terms of the first fundamental quantities and derivatives (if n>2), but in all cases irrationally.

Using (43), the author has calculated irrational values of  $K_1$  when n is greater than two; but the notation is so complicated that the presentation of the results seems impracticable, if not also useless,  $\dagger$ 

If  $2\nu + 1 = n$ , the sum reduces to a single term and formulas (34) and (42)

<sup>\*</sup>Cf. BOCHER, Introduction to Higher Algebra, § 11.

<sup>†</sup> In a recent paper the author has calculated the value of  $K_1$  as well as of the other curvatures of odd subscript, for a space of n-1 dimensions defined in  $R_n$  by the equation  $U(x_1\cdots x_n)=0$ . These values involve only the coefficients of the first fundamental form of  $R_n$  and their derivatives, together with the function U.

give a rational value for  $K_n^2$ ,

(44) 
$$K_n^2 = \beta^4 \Delta^2 = \frac{\epsilon^{n+2}}{n(n-1)} \Big( (fc)'(fa)^2 \cdots (fa)^n \Big) \Big( (\phi c)'(\phi b)^2 \cdots (\phi b)^n \Big) \begin{vmatrix} (f) & (f'\phi) \\ (\phi'f) & (\phi) \end{vmatrix}.$$

By the method used in K.-G. C. (p. 86), this may be reduced to Maschke's form (31):\*

$$(45) K_n^2 = \frac{1}{n \lceil (n-1)! \rceil^{n+2}} \Big( (fc)'(fa)^2 \cdots (fa)^n \Big) \Big( (\phi c)'(\phi b)^2 \cdots (\phi b)^n \Big) (f'\phi'f) (f^2\phi^2\phi).$$

The rather unsatisfactory results of this section are then as follows:

If n is odd,  $K_n^2$  is expressed by (45) as a rational function of the first fundamental quantities and their derivatives. Equations (34) and (42) give irrational expressions for the curvatures of odd index except  $K_1$ , for which no expression is here given.

### PART II.

## Invariants of $R_{\lambda}$ in $R_{n}$ .

The quantities  $K_{2\nu}$  and  $K_n^2$ , for n odd, are by their forms (40) and (45) differential invariants of the first fundamental quadratic form (3). When (3) defines the arc-element of a space  $R_n$  of n dimensions contained in an euclidean space  $S_{n+1}$  of n+1 dimensions, these K's have the geometric meaning already assigned to them. It is our object  $\dagger$  to find corresponding invariants of a space  $R_{\lambda}$  of  $\lambda$  dimensions, represented as differential parameters of a general space  $R_n$  of higher dimensions containing  $R_{\lambda}$ .

## § 1. Definitions and Preliminary Formulas.

In the general space  $R_n$ , of n dimensions, whose coördinates are  $x_1, \dots, x_n$  and whose arc-element is defined by equation (3), let the space  $R_{\lambda}$  of  $\lambda$  dimensions ( $\lambda < n$ ) be defined by the  $n - \lambda$  equations

(46) 
$$U^{\lambda+1}(x_1, \dots, x_n) = \text{const.}, \dots, U^n(x_1, \dots, x_n) = \text{const.}$$

If  $\lambda$  other arbitrarily chosen functions of  $x_1, \dots, x_n$ , say  $u', \dots, u^{\lambda}$ , such that

$$\Delta = (u' \cdots u^{\lambda} U^{\lambda+1} \cdots U^n) \neq 0,$$

are adjoined to these, the space  $R_{\scriptscriptstyle \lambda}$  may also be represented in parametric form

(47) 
$$x_1 = x_1(u', \dots, u^{\lambda}), \dots, x_n = x_n(u', \dots, u^{\lambda}),$$

<sup>\*</sup> In MASCHKE's reduction there are two slight numerical errors which balance each other. His equation (30) differs from (44) above in that he has divided by  $n^2$  instead of by n(n-1); while in his reduction of (30) there are n-1 of the terms which become equal, instead of n.

<sup>†</sup> Cf. K.-G. C., § 5.

by solving the  $n - \lambda$  equations (46) with the  $\lambda$  equations

$$u'(x_1, \dots, x_n) = u', \dots, u^{\lambda}(x_1, \dots, x_n) = u^{\lambda}.$$

Any n differentials satisfying the  $n - \lambda$  equations, found by differentiating (46),

$$\sum_{i=1}^{n} U_{i}^{\lambda+1} dx_{i} = 0, \dots, \quad \sum_{i=1}^{n} U_{i}^{n} dx_{i} = 0$$

determine a certain direction in  $R_{\lambda}$ . In order to find these differentials in terms of  $du', \dots, du^{\lambda}$ , we differentiate also equations (48) and solve the set

$$u_1' dx_1 + \cdots + u_n' dx_n = du',$$
  
 $u_1' dx_1 + \cdots + u_n' dx_n = du^{\lambda},$   
 $U_1^{\lambda+1} dx_1 + \cdots + U_n^{\lambda+1} dx_n = 0,$   
 $\vdots$   
 $U_1^n dx_1 + \cdots + U_n^n dx_n = 0.$ 

If  $A^{kr}$  be the cofactor of  $u_r^k$  in  $\Delta$ , then

$$dx_{r} = \frac{1}{\Delta} \sum_{k=1}^{\lambda} A^{kr} du^{k}$$

and therefore,

(49) 
$$\sum_{r=1}^{n} p_{r} dx_{r} = \frac{1}{\Delta} \sum_{k=1}^{\lambda} \left\{ u' \cdots u^{k-1} p u^{k+1} \cdots u^{\lambda} U \right\} du^{k},$$

where p is any ordinary function of  $x_1, \dots, x_n$ .

In order to find the expression for ds in terms of  $u', \dots, u^{\lambda}$ , we introduce for the differential quantic (3) the symbolic form

$$ds^2 = \sum_{i,k}^{1,\ldots,n} a_{ik} dx_i dx_k = \left[ \sum_{i=1}^{n} f_i dx_i \right]^2.$$

Then (49) gives for the length element in  $R_{\lambda}$ 

(50) 
$$ds^{2} = \frac{1}{\Delta^{2}} \left[ \sum_{i=1}^{\lambda} \left\{ u' \cdots u^{i-1} f u^{i+1} \cdots u^{\lambda} U \right\} du_{i} \right]^{2}$$
$$= \frac{1}{\beta^{2} \Delta^{2}} \left[ \sum_{i=1}^{\lambda} \left( u' \cdots u^{i-1} f u^{i+1} \cdots u^{\lambda} U \right) du_{i} \right]^{2}.$$

We may also introduce for  $ds^2$ , as given in terms of  $u', \dots, u^{\lambda}$ , the symbolic form

(51) 
$$ds^2 = \left[\sum_{i=1}^{\lambda} f_i du^i\right]^2.$$

By comparing (50) and (51) we find

(52) 
$$f_i = \frac{1}{\Delta} \{ u' \cdots u^{i-1} f u^{i+1} \cdots u^{\lambda} U \} = \frac{1}{\beta \Delta} (u' \cdots u^{i-1} f u^{i+1} \cdots u^{\lambda} U).$$

If we use the symbols of form (51), the invariants  $K_{2\nu}$  and  $K_{\lambda}^2$  ( $\lambda$  odd) of  $R_{\lambda}$  may be written, by (40) and (45),

$$(53) \quad (2\nu)! (\lambda - 2\nu)! [(\lambda - 1)!]^{\nu} K_{2\nu} = G_{2\nu} = ((\mathfrak{f}\mathfrak{a})' \cdots (\mathfrak{f}\mathfrak{a})^{2\nu} \mathfrak{f}^{2\nu+1} \cdots \mathfrak{f}^{\lambda}) (\mathfrak{f}' \cdots \mathfrak{f}^{\lambda}),$$

$$\lambda [(\lambda - 1)!]^{\lambda+2} K_{\lambda}^{2} = G_{\lambda}^{2} = ((\mathfrak{f}\mathfrak{c})' (\mathfrak{f}\mathfrak{a})^{2} \cdots (\mathfrak{f}\mathfrak{a})^{\lambda})$$

$$\times ((\mathfrak{g}\mathfrak{c})' (\mathfrak{g}\mathfrak{b})^{2} \cdots (\mathfrak{g}\mathfrak{b})^{\lambda}) (\mathfrak{f}'\mathfrak{g}' \mathfrak{f}^{3} \cdots \mathfrak{f}^{\lambda}) (\mathfrak{f}^{2}\mathfrak{g}^{2} \cdots \mathfrak{g}^{\lambda}),$$

where  $G_{2\nu}$  and  $G_{\lambda}^2$  are introduced merely for convenience. In all invariantive brackets containing the new symbols, of the quadratic form (51), the differentiation is with respect to the  $\lambda$  variables  $u', \dots, u^{\lambda}$ . This is indicated sufficiently by the German type and the number of symbols inside the brackets.  $\beta_{\mu}$  is defined by the equation

$$(f' \cdots f^{\lambda}) = \beta_{\mu} \{ f' \cdots f^{\lambda} \}.$$

We now proceed to compute the values of the invariantive expressions used in (53) and (54) in terms of the symbols of the first fundamental form (3), of  $R_n$  and the functions  $U^{\lambda+1}$ , ...,  $U^n$  which define  $R_{\lambda}$  in  $R_n$ .

By means of (52) and D. P. (3), we obtain

$$\{\mathfrak{f}'\cdots\mathfrak{f}^{\lambda}\}=rac{1}{\Delta^{\lambda}}\{f'\cdots f^{\lambda}U\}\{u'\cdots u^{\lambda}U\}^{\lambda-1}=rac{1}{\Delta}\{f'\cdots f^{\lambda}U\},$$

so that

(55) 
$$\frac{1}{\beta_{\cdot \cdot}}(\mathfrak{f}' \cdots \mathfrak{f}^{\lambda}) = \frac{1}{\beta \Delta}(f' \cdots f^{\lambda} U).$$

To calculate the value of  $\beta_u$ , square (55) and simplify the result by placing  $(f' \cdots f^{\lambda})^2 = \lambda!$ , according to M. I. (17), and  $(f' \cdots f^{\lambda}U)^2 = \lambda! (n-\lambda)! \Delta^{n-\lambda}U$  by (38). This gives

(56) 
$$\beta_u = \omega \beta \Delta, \qquad \omega = \sqrt{\frac{1}{(n-\lambda)! \Delta^{n-\lambda} U}}.$$

Then

$$(\mathfrak{f}'\cdots\mathfrak{f}^{\lambda})=\omega(f'\cdots f^{\lambda}U).$$

The other invariantive forms in (53) and (54) are reduced by the same method, and by interchanging equivalent symbols, giving \*

$$(\mathfrak{f}' \cdots \mathfrak{f}^{\lambda}) = \omega(f' \cdots f^{\lambda}U),$$

$$(\mathfrak{f}'\mathfrak{g}'\mathfrak{f}^{3} \cdots \mathfrak{f}^{\lambda}) = \omega(f'\phi'f^{3} \cdots f^{\lambda}U), \quad (\mathfrak{f}^{2}\mathfrak{g}^{2} \cdots \mathfrak{g}^{\lambda}) = \omega(f^{2}\phi^{2} \cdots \phi^{\lambda}U)$$

$$((\mathfrak{f}\mathfrak{a})' \cdots (\mathfrak{f}\mathfrak{a})^{2\nu}\mathfrak{f}^{2\nu+1} \cdots \mathfrak{f}^{\lambda})$$

$$= \omega(\omega(faU)', \omega(faU)^{2}, \cdots, \omega(faU)^{2\nu}, f^{2\nu+1} \cdots f^{\lambda}U),$$

<sup>\*</sup>Inside the invariantive brackets, we have followed MASCHKE's custom of omitting commas between symbols, except where ambiguity might occur. Cf. M. I., p. 448.

(57) 
$$\frac{\left((\operatorname{fc})'(\operatorname{fa})^{2}\cdots(\operatorname{fa})^{\lambda}\right)=\omega\left(\omega(fcU)',\,\omega(faU)^{2},\,\cdots,\,\omega(faU)^{\lambda}U\right),}{\left((\operatorname{gc})'(\operatorname{gb})^{2}\cdots(\operatorname{gb})^{\lambda}\right)=\omega\left(\omega(\phi cU)',\,\omega(\phi bU)^{2},\,\cdots,\,\omega(\phi bU)^{\lambda}U\right).}$$

§ 2. Expression for  $K_{2\nu}$ .

By (53) and (57),

$$G_{2\nu} = \omega^2(\omega(faU)', \dots, \omega(faU)^{2\nu}, f^{2\nu+1} \dots f^{\lambda}U)(f' \dots f^{\lambda}U).$$

Applying D. P. (4) to the second member, we get

$$G_{2\nu} = \omega^{2\nu+2} \big( (fa\,U)' \cdots (fa\,U)^{2\nu} f^{2\nu+1} \cdots f^{\lambda}\,U \, \big) (f' \cdots f^{\lambda}\,U) + \omega^{2\nu+1} ($$

$$\times \sum_{k=1}^{2\nu} (fa\, U)^k \big( (fa\, U)' \cdots (fa\, U)^{k-1}, \, \omega, (fa\, U)^{k+1} \cdots (fa\, U)^{2\nu} f^{2\nu+1} \cdots f^{\lambda} U \big).$$

It will now be shown that each term of this last sum vanishes. Aside from the factor  $\omega^{2\nu+1}$ , each odd term of this sum may be written in the form

$$T = (-1)^{\lambda-1} (f^{k+1} \cdots f^{\lambda} f' \cdots f^{k} U) (fa U)^{k}$$

$$\times ((fa U)' \cdots (fa U)^{k-1}, \omega, (fa U)^{k+1} \cdots (fa U)^{2\nu} f^{2\nu+1} \cdots f^{\lambda} U).$$

Applying D. P. (1) to the first two brackets of the second member, we obtain

$$T = (-1)^{\lambda-1} \begin{bmatrix} (f^k f^{k+2} \cdots f^{\lambda} f' \cdots f^k U)(f^{k+1} a^2 \cdots a^{\lambda} U) \\ + (a^2 f^{k+2} \cdots f^{\lambda} f' \cdots f^k U)(f^k f'^{k+1} a^3 \cdots a^{\lambda} U) \\ + (a^3 f'^{k+2} \cdots f^{\lambda} f' \cdots f^k U)(f^k a^2 f'^{k+1} a^4 \cdots a^{\lambda} U) \\ + \vdots & \vdots & \vdots & \vdots \\ + (a^{\lambda} f^{k+2} \cdots f^{\lambda} f' \cdots f^k U)(f^k a^2 \cdots a^{\lambda-1} f^{k+1} U) \end{bmatrix} \\ \times \left( (fa U)' \cdots (fa U)^{k-1}, \omega, (fa U)^{k+1} \cdots (fa U)^{2\nu} f^{2\nu+1} \cdots f^{\lambda} U \right).$$

Of these  $\lambda$  terms, the first vanishes because of two identical rows in the first bracket, while the others become equal to each other if we interchange  $f^{k+1}$  with  $a^2 \cdots a^{\lambda}$  in turn and in each case restore the original order in  $(faU)^{k+1}$  by the interchange of two rows.\* Thus

$$\begin{split} T &= (\,-1\,)^{\lambda-1}(\,1-\lambda\,)(f^{\,k+1}\,\cdots\,f^{\,\lambda}f'\,\cdots\,f^{\,k}U\,)(f^{\,k}a^2\,\cdots\,a^{\lambda}\,U\,) \\ &\quad \times \left(\,(fa\,U\,)'\,\cdots\,(fa\,U\,)^{k-1},\,\omega,\,(fa\,U\,)^{\,k+1}\,\cdots\,\right) = (\,-\,1\,)^{\lambda-1}(1-\lambda\,)\,T. \end{split}$$

Hence T=0 for odd values of k.

If k is even, each term T may be written

$$T = (f^{k-1} \cdots f^{\lambda} f' \cdots f^{k-2} U) (fa U)^{k}$$

$$\times ((fa U)' \cdots (fa U)^{k-1}, \omega, (fa U)^{k+1} \cdots (fa U)^{2\nu} f^{2\nu+1} \cdots f^{\lambda} U).$$
\*Cf. K.-G. C., p. 92.

By applying D. P. (1) to the first two brackets, and proceeding as above, one finds T=0 also for even values of k.

With the help of these results (58) becomes

$$G_{2\nu} = \omega^{2\nu+2} \big( (fa \, U)' \cdots (fa \, U)^{2\nu} f^{2\nu+1} \cdots f^{\lambda} \, U \big) (f' \cdots f^{\lambda} \, U).$$

Then, by (53) and (56),

(59) 
$$K_{2\nu} = \frac{(\lambda - 1)! ((faU)' \cdots (faU)^{2\nu} f^{2\nu+1} \cdots f^{\lambda} U) (f' \cdots f^{\lambda} U)}{(2\nu)! (\lambda - 2\nu)! [(\lambda - 1)! (n - \lambda)! \Delta^{n-\lambda} U]^{\nu+1}}.$$

If  $2\nu = \lambda$ , (59) becomes

(60) 
$$K_{\lambda} = \frac{\left( (faU)' \cdots (faU)^{\lambda} U \right) (f' \cdots f^{\lambda} U)}{\lambda \left[ (\lambda - 1)! (n - \lambda)! \Delta^{n - \lambda} U \right]^{(\lambda + 2)/2}},$$

which agrees with Maschke's form, K.-G. C. (60). The symbols f and a belong to the quadratic form (3), expressing the length element of  $R_n$ . Further,  $(faU)^i = (f^ia^2 \cdots a^{\lambda}U^{\lambda+1} \cdots U^n)$ , in which  $f^i$  is equal to  $f^i$  in  $(f' \cdots f^{\lambda}U)$ , while the sets of symbols  $a^2 \cdots a^{\lambda}$  are equal in any two consecutive brackets  $(faU)^{2k-1}$ ,  $(faU)^{2k}$  and otherwise distinct.

The result is then that  $K_{2\nu}$ , for the space  $R_{\lambda}$ , is expressible rationally in terms of the coefficients of the first fundamental form of  $R_n$  and their derivatives, together with the functions  $U^{\lambda+1}$ , ...,  $U^n$  (which define  $R_{\lambda}$  in  $R_n$ ) and their derivatives.

§ 3. Expression for 
$$K_{\lambda}^2$$
 when  $\lambda$  is odd.\*

The invariant  $K^2_{\lambda}(\lambda \text{ odd})$  can be expressed in a manner similar to the above. Substituting from (57) into (54), one gets

$$(61) \begin{array}{l} G_{\lambda}^{2} = \omega^{4} \left( \omega \left( fc \, U \right)', \, \omega \left( fa \, U \right)^{2}, \, \cdots, \, \omega \left( fa \, U \right)^{\lambda} U \, \right) \\ \times \left( \omega (\varphi c \, U)', \, \omega (\varphi b \, U)^{2}, \, \cdots, \, \omega (\varphi b \, U)^{\lambda} \, U \, \right) \left( f' \varphi' f^{3} \cdots f^{\lambda} \, U \, \right) \left( f'^{2} \varphi^{2} \cdots \varphi^{\lambda} \, U \, \right). \\ \text{By D. P. } (4), \\ \left( \omega (fc \, U)', \, \omega (fa \, U)^{2}, \, \cdots, \, \omega (fa \, U)^{\lambda} \, U \, \right) = \omega^{\lambda} \left( (fc \, U)' (fa \, U)^{2} \cdots (fa \, U)^{\lambda} \, U \, \right) \\ + \omega^{\lambda - 1} \left( fc \, U \right)' \left( \omega, \, (fa \, U)^{2} \cdots (fa \, U)^{\lambda} \, U \, \right) \\ + \omega^{\lambda - 1} \sum_{i = 2}^{\lambda} \left( fa \, U \right)^{i} \left( (fc \, U)' (fa \, U)^{2} \cdots (fa \, U)^{i - 1}, \, \omega, \, (fa \, U)^{i + 1} \cdots (fa \, U)^{\lambda} \, U \, \right) \\ \equiv \omega^{\lambda} \, \alpha_{1} + \omega^{\lambda - 1} \, \alpha_{2} + \omega^{\lambda - 1} \, \alpha_{3}. \\ \left( \omega (\varphi c \, U)', \, \omega (\varphi b \, U)^{2}, \, \cdots, \, \omega (\varphi b \, U)^{\lambda} \, U \, \right) = \omega^{\lambda} \left( (\varphi c \, U)' (\varphi b \, U)^{2} \cdots (\varphi b \, U)^{\lambda} \, U \, \right) \\ + \omega^{\lambda - 1} \left( \varphi c \, U \right)' \left( \omega, \, (\varphi b \, U)^{2} \cdots (\varphi b \, U)^{\lambda} \, U \, \right) \\ + \omega^{\lambda - 1} \sum_{k = 2}^{\lambda} \left( \varphi b \, U \right)^{k} \left( (\varphi c \, U)' (\varphi b \, U)^{2} \cdots (\varphi b \, U)^{k - 1}, \, \omega, \, (\varphi b \, U)^{k + 1} \cdots (\varphi b \, U)^{\lambda} \, U \, \right) \\ \equiv \omega^{\lambda} \, \beta_{1} + \omega^{\lambda - 1} \, \beta_{2} + \omega^{\lambda - 1} \, \beta_{3}. \\ \hline \, \text{* See K.-G. C., p. 93.} \end{array}$$

The notations  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$ ,  $\beta_1$ ,  $\beta_2$ ,  $\beta_3$  are used for brevity to represent the expressions whose relative places they occupy. If we also use

$$\gamma = (f'\phi'f^3 \cdots f^{\lambda}U), \qquad \delta = (f^2\phi^2 \cdots \phi^{\lambda}U),$$

then

(62) 
$$G_{\lambda}^{2} = \omega^{2\lambda+2} \left[ \omega \alpha_{1} + \alpha_{2} + \alpha_{3} \right] \left[ \omega \beta_{1} + \beta_{2} + \beta_{3} \right] \gamma \delta.$$

The nine terms of this product (omitting powers of  $\omega$ ) will now be considered in the following order:

1) 
$$\alpha_1 \beta_1 \gamma \delta$$
, 4)  $\alpha_3 \beta_2 \gamma \delta$ , 7)  $\alpha_1 \beta_3 \gamma \delta$ ,

2) 
$$\alpha_1 \beta_2 \gamma \delta$$
, 5)  $\alpha_2 \beta_1 \gamma \delta$ , 8)  $\alpha_3 \beta_3 \gamma \delta$ ,  
3)  $\alpha_2 \beta_2 \gamma \delta$ , 6)  $\alpha_2 \beta_3 \gamma \delta$ , 9)  $\alpha_3 \beta_1 \gamma \delta$ .

3) 
$$\alpha_{\alpha}\beta_{\alpha}\gamma\delta$$
, 6)  $\alpha_{\alpha}\beta_{\alpha}\gamma\delta$ , 9)  $\alpha_{\alpha}\beta_{\alpha}\gamma\delta$ 

For the first we have  $\alpha, \beta, \gamma \delta = L$ , where

(63) 
$$L = \left( (fcU)'(faU)^2 \cdots (faU)^{\lambda}U \right) \\ \times \left( (\phi cU)'(\phi bU)^2 \cdots (\phi bU)^{\lambda}U \right) (f'\phi'f^3 \cdots f^{\lambda}U) (f^2\phi^2 \cdots \phi^{\lambda}U).$$

The second is shown to vanish as follows:

$$2) \ \alpha_{1}\beta_{2}\gamma\delta = (f'\phi'f^{3}\cdots f^{\lambda}U)(\phi'c^{2}\cdots c^{\lambda}U) \\ \times ((fc^{T} \qquad (faU)^{\lambda}U)(f^{2}\phi^{2}\cdots \phi^{\lambda}U)(\omega, (\phi bU)^{2}\cdots (\phi bU)^{\lambda}U) \\ = \begin{bmatrix} (\phi'\phi', f'' & U)(f'c^{2}\cdots c^{\lambda}U) \\ +(c^{i}\phi', f'^{3} & f^{\lambda}U)(\phi'f'c^{3}\cdots e^{\lambda}U) \\ +(c^{3}\phi'f^{3}\cdots f^{\lambda}U)(\phi'c^{2}f'c^{4}\cdots c^{\lambda}U) \\ + & \cdot & \cdot & \cdot & \cdot \\ +(c^{\lambda}\phi'f^{3}\cdots f^{\lambda}U)(\phi'c^{2}\cdots c^{\lambda-1}f'U) \end{bmatrix} ((fcU)'(faU)^{2}\cdots (faU)^{\lambda}U)\cdots \\ = (1-\lambda)(f'\phi'f^{3}\cdots f^{\lambda}U)(\phi'c^{2}\cdots c^{\lambda}U)((fcU)'(faU)^{2}\cdots (faU)^{\lambda}U)\cdots \\ = (1-\lambda)\alpha_{1}\beta_{2}\gamma\delta.$$

Hence the second vanishes. The third and fourth are shown to vanish by applying D. P. (1) to exactly the same expressions.

For the fifth term,

5) 
$$\alpha_{2}\beta_{1}\gamma\delta = (f'\phi'f^{3}\cdots f^{\lambda}U)(f'c^{2}\cdots c^{\lambda}U)((\phi c\ U)'(\phi b\ U)^{2}\cdots (\phi b\ U)^{\lambda}U)$$

$$\times (\omega, (fa\ U)^{2}\cdots (fa\ U)^{\lambda}U)(f^{2}\phi^{2}\cdots \phi\ U)$$

$$= (\phi'f^{3}\cdots f'^{\lambda}f'\ U)(f'c^{2}\cdots c^{\lambda}U)((\phi c\ U)'(\phi b\ U)^{2}\cdots (\phi b\ U)^{\lambda}U)\cdots$$

By applying D. P. (1) to the first two forms and simplifying as for 2), we find

$$\alpha_2 \beta_1 \gamma \delta = (1 - \lambda) \alpha_2 \beta_1 \gamma \delta.$$

Hence 5) vanishes, and the sixth term is shown to vanish by applying D. P. (1) to the same forms.

For the seventh term,

7) 
$$\alpha_1 \beta_3 \gamma \delta = \sum_{k=2}^{\lambda} (\phi^k b^2 \cdots b^{\lambda} U) (f^2 \phi^2 \cdots \phi^{\lambda} U)$$

$$\times ((\phi c U)' (\phi b U)^2 \cdots (\phi b U)^{k-1}, \omega, \cdots)$$

$$\times (f' \phi' f^3 \cdots f^{\lambda} U) ((f c U)' (f a U)^2 \cdots (f a U)^{\lambda} U).$$

This sum is shown to vanish for all values of k by the method used for (58), and the vanishing of 8) follows by the same method.

For the last term,

9) 
$$\alpha_{3}\beta_{1}\gamma\delta = \sum_{i=2}^{\lambda} (faU)^{i}(f'\phi'f^{3}\cdots f^{\lambda}U)$$

$$\times \left((fcU)'(faU)^{2}\cdots (faU)^{i-1}, \omega, (faU)^{i+1}\cdots (faU)^{\lambda}U\right)$$

$$\times (f^{2}\phi^{2}\cdots \phi^{\lambda}U)\left((\phi cU)'(\phi bU)^{2}\cdots (\phi bU)^{\lambda}U\right).$$

The terms in which i>3 vanish by the methods used for (58), but the terms  $T_2$  (for i=2) and  $T_3$  (for i=3) do not vanish and require special treatment. We have  $T_2 = (f^3 \cdots f^{\lambda} f' \phi' U) (f^2 a^2 \cdots a^{\lambda} U) N$ , where

$$N = ((fcU)', \omega, (faU)^3 \cdots (faU)^{\lambda}U)(f^2\phi^2 \cdots \phi^{\lambda}U) \times ((\phi cU)'(\phi bU)^2 \cdots (\phi bU)^{\lambda}U)),$$

$$\begin{split} T_2 &= \begin{bmatrix} (f^2 f^4 & \cdots f^{\lambda} f' \phi' U) (f^3 a^2 & \cdots a^{\lambda} U) \\ + (a_2 f^4 & \cdots f^{\lambda} f' \phi' U) (f^2 f^3 a^3 & \cdots a^{\lambda} U) \\ + (a^3 f^4 & \cdots f^{\lambda} f' \phi' U) (f^2 a^2 f^3 a^4 & \cdots a^{\lambda} U) \\ + & \cdot & \cdot & \cdot & \cdot & \cdot \\ + (a^{\lambda} f^4 & \cdots f^{\lambda} f' \phi' U) (f^2 a^2 \cdots a^{\lambda - 1} f^3 U) \end{bmatrix} N \quad \text{[by D. P. (1)]} \\ &= (f^2 f^4 & \cdots f^{\lambda} f' \phi' U) (f^3 a^2 & \cdots a^{\lambda} U) N + (1 - \lambda) T_2. \end{split}$$

Hence

$$T_2 = \frac{1}{2} \left( f^2 f^4 \cdots f^{\lambda} f' \phi' U \right) \left( f^3 a^2 \cdots a^{\lambda} U \right) N.$$

If now, we interchange  $f^2$  and  $f^3$  and then restore the regular order of symbols, we get

$$\begin{split} T_2 &= -\frac{1}{\lambda} (f'\phi' f^3 \cdots f^{\lambda} U) (fa U)^2 \big( (fc U)', (fa U)^2, \omega, (fa U)^4 \cdots (fa U)^{\lambda} U \big) \\ &\qquad \times (f^3\phi^2 \cdots \phi^{\lambda} U) \big( (\phi c U)' (\phi b U)^2 \cdots (\phi b U)^{\lambda} U \big). \end{split}$$

Next,  $T_3$  may be written as  $(f^2\phi^2\cdots\phi^{\lambda}U)(f^3a^2\cdots a^{\lambda}U)\cdot P$ , where

$$\begin{split} P &= \left( (fcU)'(faU)^2, \, \omega, (faU)^4 \, \cdots \, (faU)^{\lambda}U \right) \\ &\qquad \qquad \times \, (f'\phi'f^3 \, \cdots \, f^{\lambda}U) \big( (\phi c\, U)'(\phi b\, U)^2 \, \cdots \, (\phi b\, U)^{\lambda}U \big) \cdot \\ &\qquad \qquad \qquad \left( (f'^3\phi^2 \, \cdots \, \phi^{\lambda}U) (f^2\, a^2 \, \cdots \, a^{\lambda}U) \right) \\ &\qquad \qquad + \, (a^2\phi^2 \, \cdots \, \phi^{\lambda}U) (f^3\, f^2\, a^3 \, \cdots \, a^{\lambda}U) \\ &\qquad \qquad + \, (a^3\phi^2 \, \cdots \, \phi^{\lambda}U) (f^3\, a^2f^2\, a^4 \, \cdots \, a^{\lambda}U) \\ &\qquad \qquad + \, (a^{\lambda}\phi^2 \, \cdots \, \phi^{\lambda}U) (f^3\, a^2 \, \cdots \, a^{\lambda-1}\, f^2U) \\ &\qquad \qquad = \, (f^3\phi^2 \, \cdots \, \phi^{\lambda}U) (f^2\, a^2 \, \cdots \, a^{\lambda}U) \, P + (1-\lambda) \, T_3, \end{split}$$

so that  $T_3 = -T_2$ .

Thus all nine terms in the second member of (62) vanish except the first, whence

(64) 
$$G_{\lambda}^{\,2} = \omega^{2\lambda+4} L \,. \label{eq:G_lambda}$$
 Then, by (54) and (56),

(65) 
$$K_{\lambda}^{2} = \frac{L}{\lambda \lceil (\lambda - 1)! (n - \lambda)! \Delta^{n - \lambda} U \rceil^{\lambda + 2}},$$

where L is given by (63), in which the symbols f,  $\phi$ , a, b, c belong to the quadratic form (3); the form  $(faU)^k = (f^k a^2 \cdots a^{\lambda} U^{\lambda+1} \cdots U^n)$ ; the f's (also  $\phi$ 's and c's) with same index are equal; the sets of symbols  $a^2, \dots, a^{\lambda}$  (also  $b^2, \dots, b^{\lambda}$ ) are equal in any two consecutive brackets of which the first has even index, and otherwise distinct.

Hence  $K^2_{\lambda}$  ( $\lambda$  odd), for the space  $R_{\lambda}$ , is expressible rationally in terms of the coefficients of the first fundamental form and their derivatives, together with the functions  $U^{\lambda+1}$ , ...,  $U^n$  (which define  $R_{\lambda}$  in  $R_n$ ) and their derivatives.